A Hecke Algebra Approach to p-adic Functionals

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Dedication

To my father who never doubted me, my mother who always supported me, and my sister who consistently kept me grounded.

Abstract

Unique model spaces for representations of reductive groups over p-adic fields play an integral role in the theory of automorphic forms (where for our purposes, 'unique' means the decomposition of the model space is multiplicity-free). Uniqueness facilitates precise computation of special functions in the model as in the work of Casselman, Shalika, and Shintani [10, 11, 43], and is a common feature of local components of global integral representations of L-functions, as in Godement and Jacquet [23], and Ginzburg and Rallis [22]. Here we study uniqueness of local model spaces with respect to 'universal unramified principal series' as outlined in Haines, Kottwitz, and Prasad [25]. In this program, the convolution algebra of compactly supported, Iwahori-biinvariant functions on $G(\mathbb{Q}_p)$ (the *Iwahori-Hecke algebra*, henceforth denoted as $\mathcal{H}$), provides powerful algebraic structure to the theory of p-adic group representations and allows one to simultaneously study the full unramified principal series. A recurring theme is that unique models and unique functionals on unramified principal series representations are associated to Hecke algebra modules of the form $\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$, where $\mathcal{H}_0$ is the finite Hecke algebra consisting of functions in $\mathcal{H}$ supported on the integer points in $G(\mathbb{Q}_p)$ and ε is a linear character of $\mathcal{H}_0$. Brubaker, Bump, and Friedberg [4] show that many standard unique functionals map to *left* induced $\mathcal{H}$-modules of this form, and Chan and Savin [12, 13] show that the Iwahori-fixed vectors in certain standard unique model spaces are associated to *right* $\mathcal{H}$-modules of this form.

We explore and expand this program in several ways. We provide sufficient conditions for an $\mathcal{H}$-module to be of the form $\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$, expanding the GL_n case described in [14]. We show that *left* $\mathcal{H}$-modules on the functional studied by Brubaker, Bump, and Friedberg [4] are essentially the same as the *right* $\mathcal{H}$-modules on the model side identified by Chan and Savin [12, 13]. We then classify, under certain conditions, the $\mathcal{H}$-modules which are associated to either unique modules or unique functionals. Finally, we investigate possible generalizations of this theory to finite multiplicity (but

not unique) model spaces, specifically the generalized Gelfand-Graev representations of Kawanaka [28, 29] (*generalized Whittaker models* in the program of Moeglin and Waldspurger [33]) of both $G(\mathbb{F}_q)$ and $G(\mathbb{Q}_p)$.

Contents

Chapter 1

Introduction

The aim of this document is to describe some interesting results and conjectures for the theory of model spaces for complex representations of split reductive groups over finite or p-adic fields. To ease the discussion, we restrict our (residue) characteristic to large primes.[1] Throughout this document, let G be a reductive group, $\mathbb{F}$ be a (finite or p-adic) field, and $G(\mathbb{F})$ be the $\mathbb{F}$-points of G. We assume $G(\mathbb{F})$ is split, and when no confusion is possible, we write $G = G(\mathbb{F})$. Given an irreducible representation π of G, a closed subgroup H of G, and a linear character ψ of H, we say $\mathrm{ind}_H^G \psi$ is a *unique model* for π if it contains π with multiplicity one. We call the space $\mathrm{ind}_H^G \psi$ a *model space* for G. Very roughly, there are two primary problems associated with model spaces for such a group G:

Problem 1.1. *Let π be a sufficiently nice irreducible representation of G. Identify a model space of G which contains π with multiplicity one.*

Problem 1.2. *Use a unique model space of a representation π to compute matrix coefficients, which are special values for linear functionals on the representation π.*

By embedding an irreducible representation in a unique model space, the computation of matrix coefficients reduces to computing an integral of complex-valued functions on the group G. Our primary motivation in addressing these questions comes from

[1] In particular, the field must have characteristic large enough to avoid all of the structure constants associated to the group.

number theory. Unique models facilitate numerous computations in representation theoretic problems connected to the theory of automorphic forms. For p-adic reductive groups, uniqueness of the model played a critical role in the work of Casselman on Macdonald's formula for the zonal spherical function [10] and of Shintani, Casselman and Shalika on formulas for the spherical Whittaker function [11, 43]. In the latter case, Whittaker functions are local components in unique global models. Over global fields, unique models are used in integral representations of automorphic L-functions to demonstrate that they possess Euler products, as in [41]. In this document we use Hecke algebras to study unique model spaces and classify unique model spaces with certain nice properties.

Before we address the non-Archimedean case, we discuss some conjectures for unique model spaces for Chevalley groups over finite fields in chapters 2 and 3 (see Carter [8] for details the construction of these groups). The model spaces we study here are closely related to the generalized Whittaker models of Moeglin and Waldspurger [33], and so our discussion here will inform our later work in the p-adic case. That said, the results we prove in the finite case are interesting independent of the p-adic theory. Let G be a Chevalley group over a field $\mathbb{F}_q$ of large characteristic. Let $B = TU$ be a Borel subgroup of G with unipotent radical U and torus T, and let W be the Weyl group of G. The prototypical example of a unique model space in this setting is the Gelfand-Graev representation (GGr), given by

$$\Gamma := \mathrm{ind}_U^G \chi$$

where χ is a non-degenerate character of U. In [20], Gelfand and Graev show that such a representation is multiplicity-free. For $G = GL_2(\mathbb{F}_q)$, every irreducible representation of G with dimension larger than 1 embeds into the corresponding GGr, so the Gelfand-Graev construction completely addresses our problem in this case. Unfortunately, for general G, there are higher-dimensional representations which do not embed in the corresponding GGr. By inducing characters from well-chosen smaller unipotent subgroups of G, we obtain the generalized Gelfand-Graev representations (gGGr) defined by Kawanaka in the 80's [29]. While we will give a precise description of the construction of gGGrs in Chapter 2, we briefly outline it here. Select a representative X of a unipotent orbit $\mathcal{O}$ in U. This unipotent element determines a parabolic subgroup $P_X = L_X U_X$ with $X \in U_X$ and a representation η of P_X. The *generalized Gelfand-Graev representation* associated to $\mathcal{O}$ is

$$\Gamma_{\mathcal{O}} := \operatorname{ind}_{U_X}^{G} \eta = \bigoplus_{\rho \in Z_{L_X}(X)^{\wedge}} (\dim \rho) \operatorname{ind}_{Z_{L_X}(X) U_X}^{G} \rho \otimes \eta$$

and is independent of the choice of representative $X \in \mathcal{O}$. We refer to the representations $\Gamma_{\mathcal{O},\rho} := \operatorname{ind}_{Z_{L_X}(X)U_X}^{G} \rho \otimes \eta$ as *reduced generalized Gelfand-Graev representations* (rgGGrs). If we pick the trivial subgroup as our unipotent subgroup, the associated unipotent orbit is the identity element and so the gGGr construction yields the regular representation of G, a model space which contains all irreducible representations of G with high multiplicity. Thus, every irreducible representation of G will occur in some generalized Gelfand-Graev model, although this model may not contain a unique copy of this representation. Inspired by Problem 1.1, the long term goal of this research is to show that each irreducible representation of G occurs uniquely in some reduced generalized Gelfand-Graev representation of G. In this direction, we have the following conjecture of Kawanaka.

Conjecture 1.3. *(Kawanaka) Let π be a unipotent representation of $G(\mathbb{F}_q)$. Then there is a reduced generalized Gelfand-Graev representation $\Gamma_{\mathcal{O},\rho}$ such that π occurs in $\Gamma_{\mathcal{O},\rho}$ with multiplicity 1.*

We investigate this conjecture for $G = GL_n(\mathbb{F}_q)$. In this case, the unipotent orbits are parametrized by partitions of n. By adapting a result of Andrews and Thiem [1] in the non-reduced case, we show

Theorem 1.4. *Let $\mathbb{F}_q$ be a finite field of large characteristic, and let G be any Chevalley group over $\mathbb{F}_q$ of type A_n. For each partition λ of n, let $\mathcal{O}_\lambda$ denote the unipotent orbit* *in G associated to λ, and let π_λ denote the unipotent representation corresponding to λ. Then π_λ embeds uniquely in the reduced generalized Gelfand-Graev representation $\Gamma_{O_{\lambda^T},\rho_{triv}}$.*

In chapter 4 we introduce some additional complexities of the p-adic theory. For this discussion, let G be a split, connected, reductive group over a non-Archimedean local field $\mathbb{F}$ with ring of integers $\mathfrak{o}$ with uniformizer ϖ and maximal ideal $\mathfrak{p} := \varpi\mathfrak{o}$, and assume the residue characteristic $q := \mathfrak{o}/\mathfrak{p}$ is large. Let W and $B = TU$ be defined as above. Our primary motivation for these investigations is the study of unramified principal series representations. The *unramified principal series* representations are constructed as follows. Let $\chi : T \to \mathbb{C}$ be a linear character which is trivial on $T(\mathfrak{o})$ (that is, an unramified character of T). By inflating χ, we can view χ as a character of B. The unramified principal series representation associated to χ is $i_B^G\chi$, where i denotes parabolic induction.[2] This class of representations has a special role in number theory, as the local components of any global automorphic representation are almost always unramified principal series representations. Unique functionals on unramified principal series representations play an important role in computing local factors in automorphic L-functions. Following Brubaker, Bump, and Friedberg, we use the representation theory of Hecke algebras to study the values of unique functionals on the unramified principal series representations of G.

A particularly useful object in the study of unramified principal series representations is the *universal unramified principal series* of G, as explicated by Haines, Kottwitz, and Prasad [25]. Let X_* denote the cocharacter lattice of T. There is an isomorphism $\chi_u : \mathbb{C}[X_*] \to \mathbb{C}[T/T(\mathfrak{o})]$ given by $\mu \mapsto \varpi^\mu := \mu(\varpi)$. Let $R := \mathbb{C}[X_*] \simeq \mathbb{C}[T/T(\mathfrak{o})]$ denote this ring. The *universal unramified principal series* of G is

$$i_B^G\chi_u^{-1} \simeq i_B^G R.$$

[2] This representation is generically irreducible. In the case that $i_B^G\chi$ is reducible, the irreducible factors in its composition series are also called principal series representations.

This construction is "universal" in the sense that $i_B^G\chi_u^{-1}$ specializes to any unramified principal series $i_B^G(\chi)$ by an extension of scalars. The universal unramified principal series is closely tied to the Iwahori-Hecke algebra of G. The Iwahori subgroup I of G is defined as the inverse image of a Borel subgroup in $G(\mathbb{F}_q)$ under the projection $G(\mathfrak{o})/G(\mathfrak{p}) \to G(\mathbb{F}_q)$. The *Iwahori-Hecke algebra* $\mathcal{H} = C_c(I\backslash G/I)$ is isomorphic to the Hecke algebra of the extended affine Weyl group $\widetilde{W} = W \ltimes X_*$ with parameter specialized to q. There is a vector space isomorphism $\mathcal{H} \simeq \mathcal{H}_0 \otimes R$, where $\mathcal{H}_0$ denotes the finite Hecke algebra $C_c(I\backslash G(\mathfrak{o})/I)$.[3] There is an isomorphism of right $\mathcal{H}$-modules

$$\mathcal{H} \simeq (i_B^G R)^I.$$

Using the formalism of the universal unramified principal series and building on a program of Brubaker, Bump, and Friedberg [4, 3], we investigate the class of R-valued functionals on the universal unramified principal series. These are left R-linear maps $\mathcal{L} : i_B^G R \to R$ typically arising as a matrix coefficient in a model space. For any unique R-valued functional $\mathcal{L}$, we get a left $\mathcal{H}$-action on the image of $\mathcal{L}$. Moreover, for any simple reflection s, we get a family of functional equations

$$\mathcal{L} \circ \mathcal{A}_s(\phi) = b_s(\mathcal{L})\mathcal{L}(\phi)^s$$

for $\phi \in \mathcal{M}$. We show the following.

Theorem 1.5. *There are finitely many choices for $b_s(\mathcal{L}) \in R$. The $\mathcal{H}$-action on the image of $\mathcal{L}$ is determined by $b_s(\mathcal{L})$ and a choice of ideal in R.*

We give an explicit description of the possible choices for $b_s(\mathcal{L})$ in Chapter 5, thus giving an algebraic classification of the unique functionals on the unramified principal series. This classification yields similar results to the analytic approach of Ginzburg [21] or the geometric approach of Sakellaridis and Venkatesh [38, 39].

[3] The finite Hecke algebra is isomorphic to the Hecke algebra of $B(\mathbb{F}_q)$-invariant functions on $G(\mathbb{F}_q)$.

The *left* $\mathcal{H}$-modules arising from unique functionals are closely related to *right* $\mathcal{H}$-modules associated to the convolution action of $\mathcal{H}$ on certain model spaces. In particular, comparing results of Chan and Savin [13] and Brubaker, Bump, and Friedberg [4], one sees that the $\mathcal{H}$ modules arising from the Whittaker model and Whittaker functional are both of the form $\operatorname{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon_{sgn}$. This theme emanates in the penumbra of the theory; the common unique local models and unique local functionals on the unramified principal series tend to produce $\mathcal{H}$-modules of the form $\operatorname{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$ for some linear character ε of $\mathcal{H}_0$. Drawing on arguments from a correspondence with Savin [40], we show

Theorem 1.6. *Assume $(\operatorname{ind}_H^G \psi)^I \simeq R$ as a right R-module under convolution. The functor $\mathcal{J}_{H,\psi}$ of twisted H, ψ coinvariants establishes an R-valued functional $\mathcal{L} : i_B^G R \to R$ which satisfies $\mathcal{L}(h\phi) = \psi(h)\mathcal{L}(\phi)$.*

This result gives us a new uniqueness criterion for functionals on the unramified principal series.

Corollary 1.7. *Let G be a reductive group over a p-adic field. Let H be a closed subgroup of G which is exhausted by its compact open subgroups, and let ψ be a smooth character of H. Assume there is an isomorphism of $\mathcal{H}$-modules*

$$(\operatorname{ind}_H^G \psi)^I \simeq \operatorname{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$$

for a linear character ε of $\mathcal{H}_0$. Then there is an R-valued functional $\mathcal{L}_{H,\psi} : i_B^G R \to R$ satisfying $\mathcal{L}_{H,\psi}(hf) = \psi(h)\mathcal{L}_{H,\psi}(f)$ which is unique up monomials in the center of $\mathcal{H}$.

In chapter 6 we revisit the (reduced) generalized Gelfand-Graev representations in the p-adic case. Sometimes called generalized Whittaker models, these provide model spaces for representations which do not have Whittaker models. In this situation, Kawanaka conjectures the following.

Conjecture 1.8. *(Kawanaka, [29]) Let π be any smooth, irreducible, unitary representation of G, and assume there is a generalized Gelfand-Graev representation*

$\Gamma_{O_X} := \operatorname{ind}_{U_X}^G \eta$ *such that* $W(\pi,\eta) \neq 0$. *Then there is a reduced generalized Gelfand-Graev representation* $\Gamma_{O_X,\rho}$ *such that* $\dim W(\pi, \rho \otimes \eta) = 1$.

We show that our results in the finite case directly inform the p-adic situation with the following Theorem.

Theorem 1.9. *Let* Γ_X *be a generalized Gelfand-Graev representation for* G*, and let* $\Gamma_X(\mathbb{F}_q)$ *the the analogous representation for* $G(\mathbb{F}_q)$*. As an* R*-module,* Γ_X^I *is generated by an* $\mathcal{H}_0$*-module isomorphic to* $\Gamma_X(\mathbb{F}_q)^{B(\mathbb{F}_q)}$.

Chapter 2

Generalized Gelfand-Graev Representations over Finite Fields

This chapter discusses the construction of generalized Gelfand-Graev representations for reductive groups over finite fields. For simplicity we assume G is a finite Chevalley group, although many of these arguments can be adapted to work in more generality. Let $W := \langle S \rangle$ be the Weyl group of G generated by a set S of simple reflections, and Φ, Φ^+, and Φ^- be the roots, positive roots, and negative roots, respectively. Let B be the Borel subgroup of G with this choice of positive roots, and let U be its unipotent radical. Let $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{u}$ be the Lie algebras of the above groups defined over the finite field.[1] For a root α, let e_α denote the corresponding root subspace, and let U_α be the corresponding root subgroup. Define $e_\alpha(1)$ so that $\{e_\alpha(1), e_{-\alpha}(1), [e_\alpha(1), e_{-\alpha}(1)]\}$ is a standard $\mathfrak{sl}_2$ triple,[2] and for any $x \in \mathbb{F}_q$, we define $e_\alpha(x) := xe_\alpha(1)$. Similarly, define $U_\alpha(x) := \exp e_\alpha(x)$. Fix a non-degenerate character $\chi : \mathbb{F}_q^+ \to \mathbb{C}$ and define a character ϕ of U by

$$\phi(\exp \sum_{\alpha\in\Phi^+} e_\alpha(a_\alpha)) := \prod_{\alpha\in\Delta} \chi(\alpha).$$

[1] In constructing a Chevalley group, one generally considers the Lie algebras defined as a $\mathbb{Z}$ span of a Chevalley basis. If $\mathfrak{g}_\mathbb{Z}$ denotes this Lie algebra, then $\mathfrak{g} := \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{F}_q$.

[2] Recall that $[X, Y, H]$ is a standard ($\mathfrak{sl}_2$) triple if X and Y are nilpotent, H is semisimple, and $[H, X] = 2X$, $[H, Y] = 2Y$, and $[X, Y] = H$. It is not hard to show that $\langle X, Y, H\rangle \simeq \mathfrak{sl}_2$.

Then the *Gelfand-Graev representation* (GGr) of G is $\mathrm{ind}_U^G \phi$.

The construction of the generalized Gelfand-Graev representation is similar. Given a representative X of a nilpotent $\mathrm{Ad}\, G$ orbit in $\mathfrak{g}$ and a nondegenerate character χ of the Lie algebra $\langle X \rangle$, one follows Kirillov's construction to produce nilpotent Lie subalgebra $\mathfrak{u}_{1.5} \subseteq \mathfrak{g}$ (the 1.5-subscript will be justified below) so that $\chi([\mathfrak{u}_{1.5}, \mathfrak{u}_{1.5}]) = 0$. Equivalently, we choose $\mathfrak{u}_{1.5}$ to be a maximal subalgebra such that the map ϕ extends to a character on $U_{1.5} := \exp \mathfrak{u}_{1.5}$. The generalized Gelfand-Graev representation associated to X will be $\mathrm{ind}_{U_{1.5}}^G \phi$ and will depend only on the nilpotent orbit of X. The reader should note that this construction works with little modification over p-adic fields with very little modification. In order to describe this construction in greater detail, we review the Dynkin-Kostant classification of nilpotent orbits in semisimple Lie algebras and the Kirillov orbit method.

2.1 Dynkin-Kostant Classification of Nilpotent Orbits

As an initial step in our construction, we review the Dynkin-Kostant classification of nilpotent orbits in semisimple Lie algebras. We follow Collingwood and McGovern's description of the classification of nilpotent orbits for complex semisimple Lie algebras in [16], although this discussion is equally valid over any algebraically closed field of characteristic 0 as discussed in [27]. This classification was adapted by Kawanaka to semisimple Lie algebras in positive characteristic in [28]. We begin by reviewing the theory in the classical setting; for the first part of this discussion we take $\mathfrak{g}$ to be a finite dimensional complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and adjoint group G_{ad}. Let $\{e, f, h\}$ be a standard triple, so that $\langle e, f, h \rangle \simeq \mathfrak{sl}_2(\mathbb{C})$, $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. Recall the Jacobson-Morozov Theorem, which states that, for any nilpotent $X \in \mathfrak{g}$, there is a homomorphism $\phi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ such that

$\phi(e) = X$ (equivalently, each nilpotent $X \in \mathfrak{g}$ is the *nilpositive element* in a standard triple). By theorems of Kostant and Mal'cev, if two standard triples in $\mathfrak{g}$ share a neutral or a nilpositive element, they are G_{ad} conjugate. We call the semisimple elements which can appear as neutral elements in standard triples *distinguished*. The above theorems establish a bijection between nilpotent orbits and distinguished semisimple orbits.

Let H be a distinguished semisimple element of $\mathfrak{g}$ in some standard triple $\{H, X, Y\}$ with nilpositive X. Since all Cartan subalgebras of $\mathfrak{g}$ are conjugate, we may assume without loss of generality that $H \in \mathfrak{h}$. Viewing $\mathfrak{g}$ as an $\mathfrak{sl}_2(\mathbb{C})$ module by the adjoint action of $\langle H, X, Y\rangle$, $\mathfrak{sl}_2(\mathbb{C})$ theory implies that each $\operatorname{ad} H$ eigenvalue in $\mathfrak{g}$ is integral, and possibly conjugating by an element of the Weyl group, we may assume that H is dominant. Such an H is uniquely determined by the values $\alpha(H)$ for $\alpha \in \Delta$. The assignment $\Delta \to \mathbb{Z}$ given by $\alpha \mapsto \alpha(H)$ is called the *weighted Dynkin diagram* associated to H. Since each H is conjugate to a unique dominant coroot, each semisimple orbit is associated to a unique weighted Dynkin diagram, and by construction if two distinguished semisimple elements H and H' share a weighted Dynkin diagram, they must be conjugate. We have established the following:

Theorem 2.1. *(Dynkin-Kostant Classification of Nilpotent Orbits) Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then there is a bijection*

$$\{\textit{nilpotent orbits in } \mathfrak{g}\} \leftrightarrow \{\textit{weighted Dynkin diagrams for } \mathfrak{g}\}.$$

If H is a distinguished dominant coroot, then $\alpha(H) \in \{0, 1, 2\}$ for all $\alpha \in \Delta$. In particular, there are no more than $3^{|\Delta|}$ weighted Dynkin diagrams, so the set of nilpotent orbits in any complex semisimple Lie algebra is finite. However, in general there are many fewer than $3^{|\Delta|}$ nilpotent orbits. In type A, the Jordan decomposition implies nilpotent orbits are parametrized by partitions of n, which grow like $O(e^{\pi\sqrt{2n/3}}) \ll 3^n$.[3]

[3] In fact, the nilpotent orbits are parametrized by sets of integer partitions in the other classical types, so similar asymptotic results can be stated in more generality.

Over finite fields, it is sometimes more convenient to describe the construction backwards. Given a weighted Dynkin diagram $D : \Delta \to \mathbb{Z}$, let H_D be the associated distinguished semisimple element under the above bijection, except with its entries taken over $\mathbb{F}_q$. In section 2.1 of [28], Kawanaka showed the following positive characteristic version of the Dynkin-Kostant classification:

Theorem 2.2. *(Dynkin-Kostant Classification in Positive Characteristic) Let G be a semisimple and simply connected algebraic group over an algebraically closed field of good characteristic with Lie algebra $\mathfrak{g}$ and Frobenius map σ. Let D be a weighted Dynkin diagram and let H_D be the σ-fixed distinguished coroot corresponding to D. Let $L := \langle \exp g \mid g \in \mathfrak{g}, [H_D, g] = 0\rangle$. Then $\mathrm{Ad}(L)$ has a Zariski-dense orbit in $\mathfrak{u}_2 := \{u \in \mathfrak{u} \mid [H_D, u] = 2u\}$. Taking X_D in that orbit, define $O_{X_D} := \mathrm{Ad}(G)X_D$. Then the pairing $D \leftrightarrow O_{X_D}$ defines a bijection*

$$\{\textit{Weighted Dynkin Diagrams for } G\} \leftrightarrow \{\sigma\textit{-stable nilpotent } Ad(G)\textit{-orbits}\}.$$

By a result of Springer (Theorem 4.33 of [44]), in large characteristic there is a bijection between the nilpotent orbits in $\mathfrak{g}$, the unipotent orbits in G, and the nilpotent orbits in the complex semisimple Lie algebra of the same type as $\mathfrak{g}$.

2.1.1 Dynkin-Kostant Theory for $GL_n(\mathbb{F}_q)$

In preparation for our later discussions of gGGrs for $G = GL_n(\mathbb{F}_q)$, we will give an explicit account of the Dynkin-Kostant classification in this case. By the Jordan decomposition, unipotent orbits in G are given by Jordan blocks. Thus each nilpotent orbit is associated to a unique partition of n, where the nilpotent orbit corresponding to $\lambda = [\lambda_1, \ldots, \lambda_r]$ has 1's on the superdiagonal of each block of size λ_i. For example, if

$n = 6$ and $\lambda = [3, 2, 1]$, the corresponding nilpotent orbit will have

$$\left(\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

as a representative. For a partition λ of n, let X_λ denote the nilpotent element chosen in this way. In order to identify the associated weighted Dynkin diagram, we must first identify a distinguished semisimple element H_λ which is the neutral element in some standard triple having X_λ as a nilpositive element. To this end, for each $k \in \mathbb{N}$, define $H_k := \mathrm{Diag}(k, k-2, \ldots, -k+2, -k)$. Then $[H_k, X_{[k]}] = 2X_{[k]}$ by construction. Applying this by blocks, we see that if $H_\lambda = \mathrm{Diag}(H_{\lambda_1}, \ldots, H_{\lambda_r})$, then $[H_\lambda, X_\lambda] = 2X_\lambda$. While not necessary for this construction, the corresponding Y_λ so that $\{X_\lambda, Y_\lambda, H_\lambda\}$ is a standard triple can be computed directly by insisting that in each block of size k, $\{X_k, Y_k, H_k\}$ generates the irreducible representation of $\mathfrak{sl}_k(\mathbb{F}_q)$ with dimension $k + 1$.

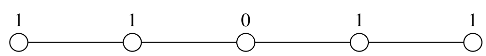
For any coroot H, we have a grading of $\mathfrak{g}$ by H-eigenvalues, that is, $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$, where $\mathfrak{g}_i$ is the i-eigenspace of $\mathrm{ad}\, H$. In constructing the generalized Gelfand-Graev representation, we study the following Lie algebras:

$$\mathfrak{u}_i := \bigoplus_{i\geq k} \mathfrak{g}_i$$

Let U_k denote $\langle \exp(\mathfrak{u}_k)\rangle$. Unfortunately, H_λ as defined above is not a dominant coroot with respect to the standard choice of positive roots for GL_n, and this makes describing the H_λ-eigenspaces $\mathfrak{g}_i$ somewhat tricky. To resolve this, we choose $w_\lambda \in W$ such that $H_\lambda^{w_\lambda}$ is a dominant coweight, and we define $\tilde{H}_\lambda := H_\lambda^{w_\lambda}$ and $\tilde{X}_\lambda := X_\lambda^{w_\lambda}$. The related weighted Dynkin diagrams can then be read off directly. In our earlier example with the $[3, 2, 1]$ partition, we find

$$H = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \tilde{H} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

The corresponding weighted Dynkin diagram is



2.2 The Kirillov Orbit Method

Dynkin-Kostant theory allows us to associate a unipotent subgroup U_1 to a nilpotent orbit by the above construction. As the next step in the construction of a generalized Gelfand-Graev representation, we use a procedure from the Kirillov orbit method to produce an irreducible representation on U_1 from a functional of $\mathfrak{u}_1$ supported on the $\mathbb{F}_q$ span of our chosen nilpotent element X viewed as an element of $\mathfrak{g}$. In its most ubiquitous form, the Kirillov orbit method describes how to construct all unitary representations of connected and simply connected nilpotent real Lie groups. We review the Kirillov orbit method in a Lie group setting before discussing positive characteristic versions in general and (unipotent subgroups of) $GL_n(\mathbb{F}_q)$ in particular.

Let U be a real, nilpotent, connected and simply connected Lie group with Lie algebra $\mathfrak{u}$. Then if $u \in U$ and $\gamma \in \mathfrak{u}^*$, the *coadjoint action* of u on γ is defined by

$$\langle u \cdot \gamma, Y \rangle := \left\langle \gamma, \mathrm{Ad}(u^{-1})Y \right\rangle,$$

where $\langle\ ,\ \rangle : \mathfrak{u}^* \times \mathfrak{u} \to \mathbb{R}$ is the canonical evaluation pairing. Given a representative γ of some coadjoint orbit in $\mathfrak{u}^*$, a *maximal subordinate subalgebra* of $\mathfrak{u}$ with respect to γ is a maximal subalgebra $\mathfrak{m} \subset \mathfrak{u}$ such that $\gamma([\mathfrak{m}, \mathfrak{m}]) = 0$. By construction, this is a maximal subalgebra such that the map $\rho : \langle \exp \mathfrak{m} \rangle \to \mathbb{C}$ given by $\rho(\exp X) := e^{\langle \gamma, X \rangle}$ is a linear character. This follows by the Baker-Campbell-Hausdorff formula, since if $X, Y \in \mathfrak{m}$,

$$\rho(\exp(X + Y)) = e^{\langle \gamma, X \rangle + \langle \gamma, Y \rangle + \langle \gamma, [X,Y] \rangle + \ldots} = e^{\langle \gamma, X \rangle} e^{\langle \gamma, Y \rangle}.$$

We then have a representation $\pi_\gamma := \operatorname{ind}_{\langle \exp \mathfrak{m} \rangle}^{U} \rho$ of U. Moreover, the following theorem holds.

Theorem 2.3. *(Kirillov, [31]) The representation π_γ defined above is an irreducible, unitary representation of U which depends only on the coadjoint orbit of γ in $\mathfrak{u}^*$. In particular, it is independent of the choice of* $\mathfrak{m}$.

This result holds in much more generality than stated. In 1965, Moore showed that Kirillov's orbit method holds over p-adic fields [34], and in 1977 Kazhdan showed a version for finite groups over fields of large characteristic [30]. A concise statement and proof of Kirillov's orbit method for unipotent groups over finite fields is presented by Panov in [35]. In particular, the field must have large characteristic, and the "unitary" criterion is vacuous.

In our later discussions, the nilpotent group in question will be a unipotent subgroup $U' \subseteq U \subseteq GL_n(\mathbb{F}_q)$ generated by root subgroups. For this discussion, we focus on the finite unipotent Lie algebra $\mathfrak{u}_1 := \mathfrak{u}_1(\mathbb{F}_q)$. One can define this as a subalgebra of $\mathfrak{g}(\mathbb{F}_q)$, or algebraically define it from the unipotent subgroup U_1 as in [35]. In particular, in large characteristic the matrix exponential will be given as a finite sum on unipotent elements. Fix a nondegenerate character $\chi : \mathbb{F}_q \to \mathbb{C}$. If U' is such a unipotent subgroup with Lie algebra $\mathfrak{u}' = \langle e_\alpha \mid \alpha \in S \subseteq \Phi^+ \rangle$, then we identify $\mathfrak{u}'^*$ with $\mathfrak{u}'^- := \langle e_{-\alpha} \mid \alpha \in S \subseteq \Phi^+ \rangle$ by composing χ with the Killing form on $\mathfrak{gl}(n, \mathbb{F}_q)$:

$$\kappa \langle X, Y \rangle = 2n \operatorname{Tr}(XY) - 2 \operatorname{Tr}(X) \operatorname{Tr}(Y).$$

If X or Y is in $\mathfrak{u}^+ \cup \mathfrak{u}^-$, the above expression reduces to $2n\,\mathrm{Tr}(XY)$. To ease notation, we normalize so that $\langle X, Y\rangle := \mathrm{Tr}\,XY$ for $X, Y \in \mathfrak{u} \cup \mathfrak{u}^-$. If $X := \sum_{\alpha\in S} e_\alpha(x_\alpha) \in \mathfrak{u}$ and $Y \in \mathfrak{gl}(n, \mathbb{F}_q)$ has e_α-coordinates y_α, then

$$\mathrm{Tr}(XY) = \sum_{\alpha\in S}[e_\alpha(x_\alpha), e_{-\alpha}(y_{-\alpha})] = \sum_{\alpha\in S} x_\alpha y_{-\alpha}.$$

In particular, the coadjoint action of U' on $\mathfrak{u}'^* = \mathfrak{u}'^-$ is given by taking the adjoint action (i.e. matrix conjugation) of U' on $\mathfrak{u}'^-$ and projecting back on to the $\mathfrak{u}'^-$ space in the obvious way.

As a concrete example, consider $G = GL_4(\mathbb{F}_q)$. Letting α_1, α_2, and α_3 denote the positive simple roots, choose the unipotent subgroup $U' = \exp(e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_1+\alpha_2} + e_{\alpha_2+\alpha_3} + e_{\alpha_1+\alpha_2+\alpha_3})$. Consider the conjugation action of an arbitrary element of U' on an arbitrary element of $\mathfrak{u}'^-$:

$$\mathrm{proj}_{\mathfrak{u}'^-}\begin{pmatrix}1 & a & b & c\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ p & 0 & 0 & 0\\ w & 0 & 0 & 0\\ y & z & q & 0\end{pmatrix}\begin{pmatrix}1 & a & b & c\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}0 & 0 & 0 & 0\\ dy+p & 0 & 0 & 0\\ ey+w & 0 & 0 & 0\\ y & -ay+z & -by+q & 0\end{pmatrix}$$

Allowing a, b, c, d, and e to vary freely, we see that there are two classes of coadjoint orbits in $\mathfrak{u}'^-$: those where the y entry is nonzero and those where it vanishes. If the y entry is nonzero, $e_{-\alpha_1-\alpha_2-\alpha_3}(y)$ is a representative for the associated coadjoint orbit. Any abelian subalgebra A of $\mathfrak{u}$ will clearly satisfy $\chi(\mathrm{Tr}\,e_{-\alpha_1-\alpha_2-\alpha_3}(y)[A, A]) = 0$. By inspection, a maximal subordinate subalgebra of $\mathfrak{u}'$ with respect to this functional is $\mathfrak{m} = e_{\alpha_1} + e_{\alpha_1+\alpha_2} + e_{\alpha_1+\alpha_2+\alpha_3}$. Note that this choice is not unique; we could have also chosen the subalgebra $e_{\alpha_3} + e_{\alpha_2+\alpha_3} + e_{\alpha_1+\alpha_2+\alpha_3}$. If $M = \exp\mathfrak{m}$, we define a character ϕ on M by $\phi(\exp(e_{\alpha_1}(a) + e_{\alpha_1+\alpha_2}(b) + e_{\alpha_1+\alpha_2+\alpha_3}(c)) = e^{\chi(yc)}$. The resulting irreducible representation of U' is then

$$\mathrm{ind}_M^{U'}\phi.$$

There are $q-1$ representations in of this kind, each one having dimension q^2. On the other hand, if $y = 0$, $[\mathfrak{u}', \mathfrak{u}'] = e_{\alpha_1+\alpha_2+\alpha_3}$ implies $\mathfrak{u}'$ is already a subordinate subalgebra to the associated functional. As a result, the irreducible unitary representation on U' will be a character whenever $y = 0$. It is noteworthy that a similar computation can be used to classify the irreducible representations of any Heisenberg group, providing a proof of the Stone-von Neumann Theorem over finite fields of large characteristic.

2.3 The Generalized Gelfand-Graev Construction

In this section, we describe Kawanaka's construction of generalized Gelfand-Graev representations and refined generalized Gelfand-Graev representations from [28] and [29]. For this section, assume G is a finite Chevalley group over $\mathbb{F}_q$ with Lie algebra $\mathfrak{g}$. First, one selects a nilpotent orbit $\mathcal{O}$ for G, and a representative X of this orbit by Dynkin-Kostant theory. By the Jacobson-Morozov Theorem (which holds for good primes by work of Pommerening [36]) X embeds in an $\mathfrak{sl}_2$ triple $\{H, X, Y\}$ in $\mathfrak{g}$. Assuming large characteristic, (say, larger than twice the number of roots), we have a grading $\mathfrak{g} = \bigoplus_{-q/2 \le i \le q/2} \mathfrak{g}_i$ of $\mathfrak{g}$ with respect to the coroot H. We define $\mathfrak{u}_k := \bigoplus_{k \le i \le q/2} \mathfrak{g}_i$ and $U_k := \langle \exp \mathfrak{u}_k \rangle$. Then $\mathfrak{u}_1$ is a nilpotent Lie algebra of $\mathfrak{g}$. Pairing X with its dual vector $X^\vee$ by the Killing form, we apply the Kirillov construction to $\mathfrak{u}_1 \otimes_{\mathbb{Z}} \mathbb{F}_q$ with the functional $\chi \circ X^\vee$ for some nontrivial character χ of $\mathbb{F}_q$. Let η denote the resulting representation of U_1.[4] Define the *generalized Gelfand-Graev representation* associated to $\mathcal{O}$ as

$$\Gamma_{\mathcal{O}} = \Gamma_X := \mathrm{ind}_{U_1}^G \eta.$$

Let L denote a Levi subgroup of G associated to U_1. L acts on $U_1^\wedge$ by $\ell\pi(u) = \pi(u^\ell)$

[4] Note that $U_1/\ker\eta$ is always a Heisenberg group. In particular, η is the unique representation of $U_1/\ker\eta$ with central character χ.

for $\ell \in L$. Let $Z_L([\eta])$ denote the stabilizer in L of the equivalence class of η under this action. We can then define a $Z_L([\eta])$-module structure on U_1 like so. For all $y \in Z_L([\eta])$, $[\eta] \simeq y[\eta]$ implies there is an intertwining operator $\tilde{\eta}(y) : [\eta] \to y[\eta]$. That is, $\tilde{\eta}(y)$ is defined (uniquely up to scalars) by the relation

$$\eta^y(x) = \tilde{\eta}(y)^{-1}\eta(x)\tilde{\eta}(y),$$

where $x \in U_1$. Define $\tilde{\eta}(zx) := \tilde{\eta}(z)\eta(x)$ for $z \in Z_L(X)$ and $x \in U_1$. $\tilde{\eta}$ will in general be a projective representation, but one may choose it to be a genuine representation over finite fields [29] and for a large class of Lie groups [47]. Moreover, by [29] and [47], the following holds:

Proposition 2.4. *Let the notation be as above. Then*

(a) $Z_L([\eta]) = Z_L(X)$,

(b) $Z_L([\eta])$ *is reductive, and*

(c) $\operatorname{ind}_{U_1}^G \eta = \bigoplus_{\rho \in Z_L([\eta])^\wedge} \dim\rho \operatorname{ind}_{Z_L([\eta])U_1}^G \rho \otimes \tilde{\eta}$.

We call the summands $\Gamma_{O,\rho} = \Gamma_{X,\rho} := \operatorname{ind}_{Z_L([\eta])U_1}^G \rho \otimes \eta$ *reduced Generalized Gelfand-Graev representations.* [5]

The $\tilde{\eta}$ action of $Z_L(X)$ can be difficult to study directly. To ease future computation, we slightly unpack the definition of η. In constructing η, we applied the Kirillov method to U_1 using the functional $\chi \circ X^\vee$, where $X^\vee$ is the dual vector to our chosen nilpotent element. Let $\mathfrak{u}_{1.5} \subseteq \mathfrak{u}_1$ denote the maximal subordinate subalgebra of $\mathfrak{u}_1$ with respect to $X^\vee$. Since $X \in \mathfrak{g}_2$, $\mathfrak{u}_2$ is a subordinant subalgebra to $\mathfrak{u}_1$ with respect to $X^\vee$. Thus, $\mathfrak{u}_2 \subset \mathfrak{u}_{1.5} \subset \mathfrak{u}_1$, justifying the notation.[6] If $\phi := \exp \chi \circ X^\vee$ denotes the character on $U_{1.5}$, by transitivity of induction we can also define the generalized Gelfand-Graev representation as $\operatorname{ind}_{U_{1.5}}^G \phi$, which will be easier to work with in some of the later sections.

[5] The terminology 'reduced' is orginally due to Yamashita [47], where in [29], Kawanaka simply calls these 'generalized Gelfand-Graev representations'.

[6] In fact, one can show that $[\mathfrak{u}_1 : \mathfrak{u}_{1.5}] = [\mathfrak{u}_{1.5} : \mathfrak{u}_2]$.

In certain circumstances, we can rephrase the definition of a refined gGGr to avoid a discussion of the intertwining representation of $Z_L(X)$:

Proposition 2.5. *If $U_{1.5}$ is normalized by $Z_L(X)$, then*

(a) ϕ extends to a linear character of $Z_L(X)U_{1.5}$,

(b) $\tilde{\eta} = \mathrm{ind}_{Z_L(X)U_{1.5}}^{Z_L(X)U_1} \phi$, and

(c) $\Gamma_{X,\rho} = \mathrm{ind}_{Z_L(X)U_{1.5}}^{G} \rho \otimes \phi$, where ρ and ϕ are both trivially extended to $Z_L(X)U_{1.5}$.

Proof. Note that $Z_L(X)$ stabilizes ϕ since $z \in Z_L(X)$ implies $z \in Z_L(X^\vee)$ by the invariance of the Killing form. Thus, if $A \in \mathfrak{u}_{1.5}$ and $z \in Z_L(X)$,

$$\phi(\exp(A)^z) = \exp(X^\vee(A^z)) = \exp(\langle X^\vee, \mathrm{Ad}\, z(A)\rangle) = \exp(\left\langle \mathrm{Ad}\, z^{-1}(X), A\right\rangle) = \phi(\exp A).$$

Then $Z_L(X)$ centralizes $U_{1.5}/\ker\phi$, so ϕ can be trivially extended to $Z_L(X)U_{1.5}$. For the second claim, note that $\mathrm{ind}_{Z_L(X)U_{1.5}}^{Z_L(X)U_1} \phi \simeq \tilde{\eta}$ as U_1-modules, and in the usual action of $Z_L(X)U_1$ on $\pi = \mathrm{ind}_{Z_L(X)U_{1.5}}^{Z_L(X)U_1} \phi$ we have

$$\eta^z(u) = \pi(u^z) = \pi(z)^{-1}\eta(u)\pi(z).$$

In particular, π satisfies the defining relations of $\tilde{\eta}$. Combining this with the 'push/pull formula' for representations (Corollary 4.3.8 of [46]) implies

$$\rho \otimes \tilde{\eta} = \rho \otimes \mathrm{ind}_{Z_L(X)U_{1.5}}^{Z_L(X)U_1} \phi = \mathrm{ind}_{Z_L(X)U_{1.5}}^{Z_L(X)U_1}((\mathrm{res}_{Z_L(X)U_{1.5}}^{Z_L(X)U_1} \rho) \otimes \phi) = \mathrm{ind}_{Z_L(X)U_{1.5}}^{Z_L(X)U_1} \rho \otimes \phi.$$

The final claim follows by transitivity of induction. □

The proof of the previous proposition is only valid for finite groups, and any generalization to general algebraic groups would require a careful understanding of $\tilde{\eta}$ as a projective $Z_L[\eta]$-module. It turns out that if $G = GL_n(\mathbb{F}_q)$, $U_{1.5}$ is always normalized by $Z_L(X)$, and it takes a very specific form. The following result is alluded to in [1], but is easy to show with our current framework.

Proposition 2.6. *Let $G = GL_n(\mathbb{F}_q)$, let λ be a partition of n, and let X be a nilpotent element in the orbit corresponding to λ. Then in our standard construction we may choose $U_{1.5}$ to be the unipotent radical of P_{λ^T}, the parabolic subgroup with a Levi subgroup L_{λ^T} consisting of block diagonal matrices with block sizes given by a composition of n obtained by reordering λ^T.*

Proof. Let $\lambda = [\lambda_1, \dots, \lambda_r]$ with $\lambda_1 \geq \cdots \geq \lambda_r$, and let $H = \mathrm{Diag}(H_{\lambda_1}, \dots, H_{\lambda_r})$ denote the standard neutral element in the $\mathfrak{sl}_2$ triple containing X_λ as a nilpositive (where $H_k = \mathrm{Diag}(k, k-2, \dots, -k)$). Let μ_i denote the number of entries of H equal to $\lambda_1 - 2i + 2$ or $\lambda_1 - 2i + 3$. Then $[\mu_1, \dots, \mu_{\lambda_1}]$ is a composition of n. We claim that μ is a reordering of the partition λ^T. Indeed, if λ has only one part, these are clearly equal. If λ has more than one part, then removing the largest part from λ decreases the size of each part of μ and and each part of λ^T by 1. By induction on the number of part sizes, we conclude that $\mu = \lambda^T$. We now show that we may choose $U_{1.5} = U_\mu$

Recall that there exists some $w \in W$ such that $\tilde{H} = H^w$ is dominant. Consider some $I + E_{i,j} \in U_2$. By definition, $\tilde{H}_{i,i} - \tilde{H}_{j,j} = 2$, so $I + E_{i,j} \in U_2$ implies $I + E_{i,j} \notin L_\mu$, meaning $U_2 \subseteq U_\mu$. Moreover, $[E_{i,j}, E_{k,l}] \in \mathrm{supp}(\phi)$ implies $i \neq k$, $j \neq l$, and either $H_{i,i} - H_{l,l} = 2$ or $H_{k,k} - H_{j,j} = 2$ (where $\tilde{H} = \mathrm{Diag}(H_{1,1}, \dots, H_{n,n})$). It follows from the construction that exactly one of $E_{i,j}$ or $E_{k,l}$ is in U_μ. In particular, the Lie algebra $\mathfrak{u}_\mu$ of U_μ is a subalgebra of $\mathfrak{u}_1$ which is subordinate ϕ. By equation 3.1.8 of [28], $[U_1 : U_{1.5}] = [U_{1.5} : U_2]$. In particular, since $\mathfrak{u}_1/\mathfrak{u}_2$ is generated by root subspaces with exactly half chosen in U_μ, we have $\dim \mathfrak{u}_\mu/\mathfrak{u}_2 = \dim \mathfrak{u}_1/\mathfrak{u}_\mu$. Thus, any maximal subordinate subalgebra containing $\mathfrak{u}_\mu$ must be equal to $\mathfrak{u}_\mu$, implying the latter can be taken as $\mathfrak{u}_{1.5}$.

□

To illustrate the choice of $U_{1.5}$ described in the above proposition, consider $G =$

$GL_7(\mathbb{F}_q)$ with $\lambda = [3, 2^2]$. Then we may choose

$$\tilde{H} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \quad \text{and} \quad U_{1.5} = \left(\begin{array}{c|ccc|ccc} 0 & * & * & \circledast & * & * & * \\ \hline 0 & 0 & 0 & 0 & \circledast & * & * \\ 0 & 0 & 0 & 0 & * & \circledast & * \\ 0 & 0 & 0 & 0 & * & * & \circledast \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

where the circled entries denote the support of $X^\vee$. As suggested in the proposition, $U_{1.5}$ is the unipotent radical of $P_{[1,3,3]}$. As $L = L_1 \subset L_{1.5} = L_{[1,3,3]}$, we indeed have that $Z_L(X)$ normalizes $U_{1.5}$.

We conclude this section with some examples of the gGGr construction for $G = GL_n(\mathbb{F}_q)$. Consider $G = GL_3(\mathbb{F}_q)$, and let α_1 and α_2 be the standard positive simple roots for G. In this case, there are three nilpotent orbits, parametrized by the partitions $[3], [2, 1]$, and $[1^3]$. We have seen that the gGGr construction for partition $[3]$ will give us the original Gelfand-Graev representation, and by convention $[1^3]$ yields the regular representation of G. The $[2, 1]$ partition therefore corresponds to one of the smallest nontrivial gGGrs. The nilpotent element in $\mathfrak{g} = \mathfrak{gl}_3(\mathbb{F}_q)$ associated to this partition is $X = e_{\alpha_1}(1)$, and the associated coroot is $H = \mathrm{Diag}(1, -1, 0)$. To have $\mathfrak{u}_1 \subset \mathfrak{u}^+$ with the usual choice of positive roots, we replace H with $\tilde{H} = \mathrm{Diag}(1, 0, -1) = H^{s_2}$. Then $\tilde{X} = e_{\alpha_1}(1)^{s_2} = e_{s_2(\alpha_1)}(1) = e_{\alpha_1+\alpha_2}(1)$. Examining the $\mathrm{Ad}\,\tilde{H}$-eigenspaces, we find $\mathfrak{u}_1 = \mathfrak{u} = e_{\alpha_1} \oplus e_{\alpha_2} \oplus e_{\alpha_1+\alpha_2}$, and $\mathfrak{u}_2 = e_{\alpha_2}$. Fixing a character $\chi : \mathbb{F}_q \to \mathbb{C}$, the functional associated to $\tilde{X}$ is $\chi \circ \tilde{X}^\vee(e_{\alpha_1}(a) + e_{\alpha_2}(b) + e_{\alpha_1+\alpha_2}(c)) = \chi(c)$.

By Proposition 2.6, we can take $\mathfrak{u}_{1.5} = e_{\alpha_1} \oplus e_{\alpha_1+\alpha_2}$ as the maximal subordinate subalgebra for the above functional when applying the Kirillov construction to U_1. If ϕ is the associated character of $U_{1.5}$, the representation $\eta = \mathrm{ind}_{U_{1.5}}^{U_1} \phi$ is the unique q dimensional representation of the Heisenberg group with central character ϕ. We obtain the generalized Gelfand-Graev representation $\mathrm{ind}_{U_1}^G \eta$. Here, $Z_L(X) = Z_L([\eta])$ consists of all matrices of the form $\mathrm{Diag}(s, t, s)$ for $s, t \in \mathbb{F}_q^\times$, so we have the decomposition

$$\mathrm{ind}_{U_1}^G \eta = \bigoplus_{\rho \in (\mathbb{F}_q^\times \times \mathbb{F}_q^\times)^\wedge} \mathrm{ind}_{Z_L([\eta])U_1}^G \rho \otimes \tilde{\eta} = \bigoplus_{\rho \in (\mathbb{F}_q^\times \times \mathbb{F}_q^\times)^\wedge} \mathrm{ind}_{Z_L(X)U_{1.5}}^G \rho \otimes \phi$$

As we employ the tools of Hecke algebras in the later chapters, we will see that many of the refined generalized Gelfand-Graev representations above do not support B-fixed vectors (and therefore do not admit the structure of a module of the Hecke algebra). In this case only the refined representation corresponding to the trivial $Z_L([\eta])$ module will have dimension larger than 1 as a module of the Hecke algebra. In this way, studying the refined Gelfand-Graev representations allows us to cut out much of the 'noise' in the unrefined version.

As a slightly more complicated example of the gGGr construction, consider $G = GL_4(\mathbb{F}_q)$ with the positive simple roots α_1, α_2, and α_3. There are five nilpotent orbits here, corresponding to the five partitions of four. Of these, $[3, 1]$, $[2^2]$, and $[2, 1^2]$ will yield a construction different from both the original Gelfand-Graev representation and the regular representation. The $[3, 1]$ case is similar to the $[2, 1]$ case above, and in general the gGGrs coming from the 'subregular orbit' - the one associated to the partition $[n-1, 1]$ - have similar $\mathcal{H}_0$-module structure. However, the generalized Gelfand-Graev representation associated to $[2^2]$ is sufficiently interesting to warrant a second example. The nilpotent element with this Jordan type is $X = e_{\alpha_1}(1) + e_{\alpha_3}(1)$, and the associated coroot is $H = \mathrm{Diag}(1, -1, 1, -1)$. Conjugating by s_2, we find $\tilde{H} = \mathrm{Diag}(1, 1, -1, -1)$, and $\tilde{X} = e_{\alpha_1+\alpha_2}(1) + e_{\alpha_2+\alpha_3}(1)$. In this case,

$$\mathfrak{u}_1 = \mathfrak{u}_2 = \bigoplus_{\alpha \geq \alpha_2} e_\alpha.$$

Then $\mathfrak{u}_1 \supseteq \mathfrak{u}_{1.5} \supseteq \mathfrak{u}_2$ implies the above is also $\mathfrak{u}_{1.5}$. Fixing a character $\chi : \mathbb{F}_q \to \mathbb{C}$, the associated character ϕ of $U_{1.5}$ is

$$\phi\left(\exp\sum_{\alpha\geq\alpha_2} e_\alpha(a_\alpha)\right) = \chi(a_{\alpha_1+\alpha_2} + a_{\alpha_2+\alpha_3}).$$

Note that $\mathfrak{u}_1 = \mathfrak{u}_{1.5}$ implies $\phi = \eta$. We see that $Z_L([\eta]) = Z_L(\tilde{X}) \simeq GL_n(\mathbb{F}_q) \hookrightarrow GL_n(\mathbb{F}_q) \times GL_2(F_q)$, viewed as block diagonal matrices in $GL_4(\mathbb{F}_q)$. Then Proposition 2.4 gives

$$\operatorname{ind}_{U_1}^G \eta = \bigoplus_{\rho\in GL_2(\mathbb{F}_q)^\wedge} (\dim\rho) \operatorname{ind}_{GL_2(\mathbb{F}_q)U_1}^G \rho\otimes\eta.$$

It follows that much of the interesting structure of this generalized Gelfand-Graev representation should come from the representation theory of $GL_2(\mathbb{F}_q)$. In later sections, we will use the understanding of the Hecke algebra action on $GL_2(\mathbb{F}_q)$ representations as a stepping stone in our understanding of the Hecke algebra action on this larger representation.

Chapter 3

Unipotent Representations and gGGrs over Finite Fields

One of Kawanaka's original motivations for his generalized Gelfand-Graev construction was to provide a parametrization of irreducible representations of reductive groups over finite fields. While we discuss this general program more in the appendix, in this section we will use Kawanaka's conjectures regarding the appearance of unipotent representations in gGGrs to guide our investigations of the Hecke algebra action on the B-fixed vectors in gGGrs. We review some basic facts about the representation theory of Hecke algebras and discuss the Hecke algebra modules obtained from representations with B-fixed vectors. We then leverage this machinery in a detailed discussion of the Hecke algebra modules obtained from (refined) gGGrs of $GL_n(\mathbb{F}_q)$. These tools will allow us to provide explicit models for irreducible representations with B-fixed vectors using refined gGGrs for $G = GL_n(\mathbb{F}_q)$.

3.1 The Springer Correspondence

The Springer correspondence shows a fundamental relationship between the nilpotent orbits of a reductive Lie algebra and the representation theory of the corresponding Weyl group. Springer first established the correspondence for reductive groups over fields of large characteristic in [44] before proving it for adjoint semisimple complex Lie groups in [45]. Let G be such a group with Lie algebra $\mathfrak{g}$ and nilpotent element X in some nilpotent $\mathrm{Ad}\,G$-orbit $\mathcal{O}_X$. Let $A(X) := G^X/(G^X)^\circ$ denote the "component group" of $Z_G(X)$.[1] The group $A(X)$ is well understood. In type A it is always trivial, in types B, C, and D it is an elementary abelian 2-group.

Theorem 3.1. *(Springer) Let $\mathcal{B}_X$ denote the variety of all Borel subalgebras of $\mathfrak{g}$ containing X. Then there are commuting actions of W and $A(X)$ on $H^*(\mathcal{B}_X)$, where $H^*(\mathcal{B}_C)$ has complex coefficients in the complex case and ℓ-adic such that $\mathbb{Q}_\ell$ contains a p^{th} root of 1 if G is over a finite field of characteristic p. Then*

(a) The top degree cohomology $H^d(\mathcal{B}_X)$ decomposes as a direct sum $\bigoplus_{\mu \in A(X)^\wedge} \pi_\mu \otimes V_\mu$, where π_μ is either 0 or an irreducible representation of W on which $A(X)$ acts trivially, V_μ is a module on which W acts trivially, and

(b) any irreducible W-module is isomorphic to π_μ for a unique nilpotent orbit $\mathcal{O}_X$ and a unique $\mu \in A(X)^\wedge$.

In type A_{n-1}, we already know that there is a bijection between the nilpotent orbits of $\mathfrak{g}$ and the representations of the Weyl group S_n. In particular, these are both enumerated by partitions of n. If λ is a partition of n and X is a nilpotent element in the λ-orbit, under the Springer correspondence X will be associated to the Specht module χ_{λ^T}.

3.2 Representations of GL_n with B-fixed Vectors in gGGrs

[1] To reconcile this topological definition with our finite setting, we can define this as $G(\overline{\mathbb{F}_q})^X/G(\overline{\mathbb{F}_q})^\circ$ and take Frobenius-fixed points.

This section discusses the appearance of representations with B-fixed vectors in gGGrs over finite fields. Noting that these are exactly the 'unipotent representations in the principal series,' we adapt Kawanaka's discussion of the appearance of unipotent representations in gGGrs in [29]. For $G = GL_n$, all unipotent representations have B-fixed vectors, so our focus on the Hecke algebra perspective fits into Kawanaka's framework without modification. We outline Kawanaka's program and discuss his conjectures in the context of Deligne-Lusztig theory in the appendix. For our purposes, the following version of Kawanaka's conjecture will suffice:

Conjecture 3.2. *(Kawanaka, [29]) Given a unipotent representation π of a finite reductive group G, there is a refined generalized Gelfand-Graev representation $\Gamma_{X,\rho}$ such that π occurs in $\Gamma_{X,\rho}$ with multiplicity 1.*

In the discussion surrounding this conjecture, Kawanaka suggests that ρ will be trivial on the connected component of $Z_L(X)$, so that it can be viewed as a representation of the component group $A(X)$ in the statement of the Springer correspondence. In a way which will be made explicit in the next section, representations with B-fixed vectors biject with representations of W. If $\gamma : W^\wedge \to \{\text{irreducible representations with } B\text{-fixed vectors}\}$ denotes this bijection, one would hope that, if (X, ρ) corresponds to a representation π of W under the Springer correspondence, $\gamma(\pi)$ would embed uniquely in $\Gamma_{X,\rho}$.

In [1], Andrews and Thiem show a preliminary result about the appearance of unipotent representations in gGGrs for $G = GL_n(\mathbb{F}_q)$. Using some heavy combinatorial machinery, they proved the following:

Theorem 3.3. *(Andrews and Thiem [1]) Let $G = GL_n(\mathbb{F}_q)$ (where $\mathbb{F}_q$ has large characteristic) and Γ_λ denote the gGGr constructed from the nilpotent orbit of partition λ. For a partition ν of n, let $\pi_\nu = \gamma(\chi_\nu)$, where χ_ν is the Specht module associated to the partition ν. Then*

$$\langle \Gamma_\lambda, \pi_{\nu^T} \rangle = K_{\nu\lambda}(q)$$

where $K_{\nu\lambda}$ is the one parameter Kostka-Foulkes polynomial.

In particular, $K_{\lambda\lambda}(q) = 1$, so the above theorem implies that each unipotent representation must appear in some refined gGGr with multiplicity one. Later in this chapter, we show that this must indeed occur in the refined gGGr predicted by the Springer correspondence, i.e. the refined gGGr obtained by taking the trivial representation of $Z_L(X)$.

3.3 Representation Theory of Hecke Algebras

For this section we consider G a reductive group over $\mathbb{F}_q$ with Borel subgroup B and Weyl group W. Recall that the (finite) *Hecke algebra* $\mathcal{H}_0$ of G is the algebra of functions $C(G//B)$ with multiplication given by convolution. We use the notation $\mathcal{H}_0$ for this Hecke algebra in order to match this exposition with the later discussion of the Iwahori-Hecke algebra in the p-adic case. By the Bruhat decomposition, $G//B$ is in bijection with elements of W, so $\mathcal{H}_0 = \mathrm{span}_{\mathbb{C}}\{T_w \mid w \in W\}$, where T_w is the characteristic function of BwB. If S denotes the set of simple reflections in W, then $\mathcal{H}_0$ can also be defined in terms of generators and relations. In particular, $\mathcal{H}_0$ is the complex algebra generated by $\{T_s \mid s \in S\}$ satisfying the *braid relations*

$$\underbrace{T_sT_tT_s\dots}_{m_{s,t}\text{ factors}} = \underbrace{T_tT_sT_t\dots}_{m_{s,t}\text{ factors}} \qquad \text{where } s,t \in S$$

and the *quadratic relations*

$$(T_s - q)(T_s + 1) = 0 \qquad \text{where } s \in S.$$

From these relations or direct computation one can derive a useful multiplication rule.

For any $s \in S$ and $w \in W$,

$$T_sT_w = \begin{cases} T_{sw} & \text{if} \quad \ell(sw) > \ell(w) \\ (1-q)T_w + qT_{sw} & \text{if} \quad \ell(sw) < \ell(w) \end{cases}$$

These relations can be used to directly define a Hecke algebra from a Coxeter group. With this perspective, one views the parameter q as an indeterminate as opposed to the cardinality of a finite field. The latter perspective makes it easier to state the following extremely useful result.

Theorem 3.4. *(Tits Deformation Theorem) If $\mathcal{H}_0^q$ is the Hecke algebra associated to a Weyl group W with q a parameter and $\mathcal{H}_0$ is the same algebra with q specialized to a prime power, then $\mathcal{H}_0 \simeq \mathcal{H}_0^q$. Moreover, if $\mathcal{H}_0^1$ is the Hecke algebra with q specialized to* 1*, then $\mathcal{H}_0^1 = \mathbb{C}[W]$ by identifying T_w with w. These specializations yield a dimension preserving bijection*

$$\left\{ \begin{array}{c} \textit{irreducible} \\ \mathcal{H}_0 - \textit{modules} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{irreducible} \\ W - \textit{modules} \end{array} \right\}.$$

We adopt the convention that if χ_λ is a Specht module, $\tilde{\chi}_\lambda$ is the corresponding representation of $\mathcal{H}_0$. An illustrative example we will use in the future is the character table of the Hecke algebra associated to $GL_4(\mathbb{F}_q)$:

Representation	1	T_{s_1}	$T_{s_1s_3}$	$T_{s_1s_2}$	$T_{s_1s_2s_3}$
$\tilde{\chi}_{[1^4]}$	1	-1	1	1	-1
$\tilde{\chi}_{[2,1^2]}$	3	$q-2$	$-2q+1$	$-q+1$	q
$\tilde{\chi}_{[2^2]}$	2	$q-1$	q^2+1	$-q$	0
$\tilde{\chi}_{[3,1]}$	3	$2q-1$	q^2-2q	q^2-q	$-q^2$
$\tilde{\chi}_{[4]}$	1	q	q^2	q^2	q^3

which is indeed a q-deformation of the character table of S_4:

Representation	1	s_1	s_1s_3	s_1s_2	$s_1s_2s_3$
$\chi_{[1^4]}$	1	−1	1	1	−1
$\chi_{[2,1^2]}$	3	1	−1	0	1
$\chi_{[2^2]}$	2	0	2	−1	0
$\chi_{[3,1]}$	3	1	−1	0	−1
$\chi_{[4]}$	1	1	1	1	1

Consider an irreducible representation π of G with a B-fixed vector. Then π^B admits a nontrivial $\mathcal{H}_0$-module structure, and by Tits deformation this can be interpreted in terms of the Weyl group of G. By the discussion in chapter 4 of [19], we have the following.

Proposition 3.5. *There is a natural bijection between the irreducible representations of G with B-fixed vectors and irreducible representations of the Weyl group given by considering the space of B-fixed vectors as an $\mathcal{H}_0$-module and applying Tits deformation.*

In light of this discussion, we can rephrase our philosophy from the previous section. Let $\mathcal{S}$ denote the map from the Springer correspondence associating an irreducible W-module to a pair (X, ρ) where X is a representative of a nilpotent orbit (i.e. X is a the nilpotent element one would obtain from Dynkin-Kostant theory) and ρ is an irreducible representation of the component group $A(X)$ viewed as a representation of $Z_L(X)$. Similarly, let $\mathcal{T} : W^\wedge \to \mathcal{H}^\wedge$ denote the bijection given by the Tits deformation theorem. Then we hope to establish the following picture for refined gGGrs:

$$\begin{tikzcd} \chi \in W^\wedge \arrow[r, "\mathcal{S}"] \arrow[d, "\mathcal{T}"] & (X, \rho) \arrow[d, "gGGr"] \\ \tilde{\chi} \in \mathcal{H}^\wedge \arrow[r, hook, dashed, "\exists!"] & (\Gamma_{X,\rho})^B \end{tikzcd}$$

Note that for $G = GL_n(\mathbb{F}_q)$, the representations ρ will always be trivial, and thus a'non-refined' GL_n version of the above is implied by Theorems 3.3 and 3.4

3.4 $\mathcal{H}_0$-module Structure of gGGrs for $GL_n(\mathbb{F}_q)$

In this section we apply the machinery established so far to investigate the above picture $GL_n(\mathbb{F}_q)$. We continue our discussion of the $[2,1]$ and $[2,2]$ examples by computing spaces of B-fixed vectors and, when practical, the complete $\mathcal{H}_0$-module structure of the refined gGGr summands of these representations. Using insights from these examples, we conclude this section with an explicit solution to Kawanaka's refined conjecture for $G = GL_n(\mathbb{F}_q)$.

3.4.1 $GL_n(\mathbb{F}_q)$: The $\Gamma_{[2,1]}$ gGGr

Let $G = GL_3(\mathbb{F}_q)$. From our earlier discussion, the orbit corresponding to $[2,1]$ has $\tilde{X} = e_{\alpha_1(1)+\alpha_2(1)}$ as a representative, with the corresponding neutral element $\tilde{H} = \mathrm{Diag}(1,0,-1)$. With these choices, $U_1 = U, U_2 = U_{\alpha_1+\alpha_2}$, and $U_{1.5} = \langle U_{\alpha_1}, U_{\alpha_1+\alpha_2}\rangle$. We get the character ϕ on $U_{1.5}$ as

$$\phi(U_{\alpha_1}(x)U_{\alpha_1+\alpha_2}(z)) = \chi(z),$$

resulting in the gGGr $\Gamma_{[2,1]} = \mathrm{ind}_{U_{1.5}}^{G}\phi$. As an initial step in studying $(\Gamma_{[2,1]})^B$, we note that the Bruhat cells in G which can support B-fixed vectors for the representation Γ will be precisely the cells BwB such that $w(\alpha_1+\alpha_2)$ is not a positive root. These are exactly the cells with representatives s_1s_2, s_2s_1, and $s_1s_2s_1$. $B\backslash G/U_{1.5}$ has double coset representatives of the form $wU_{\alpha_2}(y)$ for $w \in W$ and $y \in \mathbb{F}_q$, where the double cosets $BwU_{\alpha_2}(y)U_{1.5}$ are distinct for different choices of y exactly when $w(\alpha_2)$ is not a positive root. In the event that $w(\alpha_2)$ and $w(\alpha_1+\alpha_2)$ are both negative, we will get q B-fixed vectors in the BwB cell if and only if $w(\alpha_2) \nless w(\alpha_1+\alpha_2)$. In this example, the only

Bruhat cell with this property is Bs_1s_2B. Then if $f \in \Gamma_{[2,1]}$ and $y \neq 0$,

$$\begin{aligned}\sum_{b\in B} f(b^{-1}s_1s_2U_{\alpha_2}(y)) &= \sum_{b\in U_{1.5}}\sum_{a\in\mathbb{F}_q} f(bU_{\alpha_2}(a)s_1s_2U_{\alpha_2}(y))\\ &= \sum_{b\in U_{1.5}}\sum_{a\in\mathbb{F}_q} f(bs_1s_2U_{\alpha_1}(a)U_{\alpha_2}(y))\\ &= \sum_{b\in U_{1.5}}\sum_{a\in\mathbb{F}_q} f(bs_1s_2U_{\alpha_2}(y)U_{\alpha_1}(a)U_{\alpha_1+\alpha_2}(ay))\\ &= \sum_{b\in U_{1.5}} f(bs_1s_2U_{\alpha_2}(y))\sum_{a\in\mathbb{F}_q}\phi(e_{\alpha_1+\alpha_2}(ay)) = 0.\end{aligned}$$

We conclude that the double cosets in $B\backslash G/U_{1.5}$ which support B-fixed vectors are exactly those corresponding to s_1s_2, s_2s_1, and $s_1s_2s_1U_{\alpha_2}(y)$ for $y \in \mathbb{F}_q$. There are $q+2$ of these.

Since the number of B-fixed vectors depends on q, it can be a bit tricky to compute the $\mathcal{H}_0$-module structure of $(\Gamma_{[2,1]})^B$ directly. In this case it can be illustrative to compute the $\mathcal{H}_0$-module structure of $(\Gamma_{[2,1]})^B$ via the decomposition $\bigoplus_{\rho\in Z_L(X)^\wedge}(\Gamma_{[2,1],\rho})^B$. To this end, we recall that $Z_L(e_{\alpha_1+\alpha_2}(1))$ consists of matrices of the form $A(t,s) := \mathrm{Diag}(s,t,s)$. Then the refined gGGrs are the representations $\Gamma_{X,\rho} = \mathrm{ind}_{Z_L([\phi])U_{1.5}}^G \rho\otimes\phi$. Applying the above, the space of B-fixed vectors in this representation has a basis of functions supported on the double cosets with representatives s_1s_2, s_2s_1 and $w_0U_{\alpha_2}(y)$ for $y \in \mathbb{F}_q$. The Bruhat cell Bw_0B contains only two $B\backslash G/Z_L([\phi])U_{1.5}$ double cosets, with representatives w_0 and $w_0U_{\alpha_2}(1)$, so there are only four $B\backslash G/Z_L([\phi])U_{1.5}$ double cosets which could support B-fixed vectors.

It turns out that the double coset $Bw_0U_{\alpha_2}(1)$ supports $q-1$ B-fixed vectors. To see this, consider the representation ρ of $Z_L([\phi])$ given by $\rho(A(t,s)) = \psi(s/t)$ for some nontrivial character ψ of $\mathbb{F}_q^\times$. For each such representation, there is a B-fixed vector in $\mathrm{ind}_{Z_L(\eta)U_{1.5}}^G\rho\otimes\tilde{\eta}$ given by $f(bw_0U_{\alpha_2}(1)A(t,s)u) = f(w_0)\psi(s/t)$. The only danger is that $A(s,t) \in B$ could 'move across' $w_0U_{\alpha_2}(1)$ to change the value of the function, but $A(s,t)w_0U_{\alpha_2}(1) = w_0A(s,t)U_{\alpha_2}(1) = w_0U_{\alpha_2}(s/t)A(s,t)$. In particular, we could only move $A(s,t)$ over in this way if $s/t = 1$, in which case $f(w_0U_{\alpha_2}(1)A(s,t)) = f(w_0U_{\alpha_2}(1))$. There are $q-2$ such B-fixed functions, and the corresponding spaces $(\mathrm{ind}_{Z_L(\eta)U_{1.5}}^G \rho \otimes \tilde{\eta})^B$ are one dimensional and given by the sign representation on $\mathcal{H}_0$.

We claim that $(\mathrm{ind}_{Z_L([\eta])U_{1.5}}^G \rho_{triv} \otimes \tilde{\eta})^B$ is a four dimensional space spanned by the vectors $f_{s_1s_2}, f_{s_2s_1}, f_{w_0}$, and $f_{w_0U_{\alpha_2}(1)}$ supported on the double cosets $Bs_1s_2U_{1.5}, Bs_2s_1U_{1.5}$, $Bw_0U_{1.5}$, and $Bw_0U_{\alpha_2}U_{1.5}$, respectively. Indeed, by construction these functions are linearly independent, and they support B-fixed vectors for the same reasons that their predecessors in $(\mathrm{ind}_{U_{1.5}}^G \phi)^B$ did. We normalize the functions $f_{wU_{\alpha_2}(a)}$ to take value 1 on the double coset $BwU_{\alpha_2}(a)Z_L([\eta])$. To determine the $\mathcal{H}_0$-module structure of these vectors, we could compute the traces of T_{s_1} and $T_{s_1s_2}$ and apply the character theory of Hecke algebras. However, it can be more enlightening to compute the complete matrices of the T_{s_1} and T_{s_2} actions directly. First, note that for any simple reflection s_α and $w \in W$,

$$\begin{aligned} T_{s_\alpha} * f_{wU_{\alpha_2}(y)}(x) &= \frac{1}{|B|}\sum_{g\in G} 1_{s_\alpha}(g) f_{wU_{\alpha_2}(y)}(g^{-1}x) \\ &= \frac{1}{|B|}\sum_{g\in Bs_\alpha B} f_{wU_{\alpha_2}(a)}(gx). \end{aligned}$$

In particular, $T_{s_\alpha} * f_{wU_{\alpha_2}(y)}(x)$ is supported in

$$\begin{aligned} Bs_\alpha BwU_{\alpha_2}(y)Z_L(X)U_{1.5} &= \bigcup_{a\in\mathbb{F}_q} Bs_\alpha U_\alpha(a)wU_{\alpha_2}(y)Z_L(X)U_{1.5} \\ &= \bigcup_{a\in\mathbb{F}_q} Bs_\alpha wU_{w^{-1}(\alpha)}(a)U_{\alpha_2}(y)Z_L(X)U_{1.5}. \end{aligned}$$

Alternatively,

$$\begin{aligned} Bs_\alpha BwU_{\alpha_2}(y)Z_L(X)U_{1.5} &= \bigcup_{a\in\mathbb{F}_q^\times} BU_{-\alpha}(a)wU_{\alpha_2}(y)Z_L(X)U_{1.5} + Bs_\alpha wU_{\alpha_2}(y)Z_L(X)U_{1.5} \\ &= \bigcup_{a\in\mathbb{F}_q^\times} BwU_{w^{-1}(-\alpha)}(a)U_{\alpha_2}(y)Z_L(X)U_{1.5} + Bs_\alpha wU_{\alpha_2}(y)Z_L(X)U_{1.5}. \end{aligned}$$

The first description is useful if $\ell(s_\alpha w) > \ell(w)$ (in which case $w^{-1}(\alpha)$ is positive), and the second is used when $\ell(s_\alpha w) < \ell(w)$ (in which case $w^{-1}(-\alpha)$ is positive). Applying these to the basis $\{f_{s_1s_2}, f_{s_2s_1}, f_{w_0}, f_{w_0U_{\alpha_2}(1)}\}$, we find the matrices

$$M_{s_1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & q-1 & q-1 & q-2 \end{pmatrix} \quad \text{and} \quad M_{s_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ q & 0 & q-1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for the actions of T_{s_1} and T_{s_2}, respectively. Now that these are computed, we can quickly verify the identities $(M_{s_i} - q)(M_{s_i} + 1) = 0$ and $M_{s_i}M_{s_j}M_{s_i} = M_{s_j}M_{s_i}M_{s_j}$, to confirm that this is indeed an $\mathcal{H}_0$-module. We also get the matrix

$$M_{s_1s_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ q & 0 & q-1 & -1 \\ 0 & -1 & 0 & -1 \\ q^2-q & -q+1 & q^2-2q+1 & -q+2 \end{pmatrix}$$

for $T_{s_1s_2}$. We can then compute traces directly to see that $(\mathrm{ind}_{Z_L(\eta)U_{1.5}}^G \rho_{triv} \otimes \tilde{\eta})^B \cong \tilde{\chi}_{sgn}^2 \oplus \tilde{\chi}_{std}$ (where χ_{sgn} is the sign representation and χ_{std} is the two dimensional representation of S_3). However, explicitly computing these matrices can give us more precise information about these submodules. In particular, M_{s_1} and M_{s_2} both have a unique q-eigenspace and a three-dimensional -1-eigenspace. The simultaneous two-dimensional -1-eigenspace is spanned by $f_{s_1s_2} - f_{w_0} + f_{w_0U_{\alpha_2}(1)}$ and $f_{s_2s_1} - f_{w_0U_{\alpha_2}(1)}$, illustrating that these two vectors give us two copies of the $\mathcal{H}_0$-module $\tilde{\chi}_{sgn}$. Furthermore, $f_{s_2s_1} + f_{w_0U_{\alpha_2}(0)} + (q-1)f_{w_0U_{\alpha_2}(1)}$ and $f_{s_1s_2} + qf_{w_0U_{\alpha_2}(0)}$ (the q-eigenvectors for T_{s_1} and T_{s_2}, respectively) span a 2-dimensional $\mathcal{H}_0$-module and thus form a basis for the $\mathcal{H}_0$-module $\tilde{\chi}_{[2,1]}$.

Upon seeing this computation, it seems that we could have identified the unique q-eigenvector for T_{s_1} in $(\mathrm{ind}_{U_{1.5}}^G \phi)^B$ as $f_{s_2s_1} + \sum_{a\in\mathbb{F}_q} f_{w_0U_{\alpha_2}(a)}$, since

$$\begin{aligned} T_{s_1} * \left(f_{s_2s_1} + \sum_{a\in\mathbb{F}_q} f_{w_0U_{\alpha_2}(a)} \right) &= \sum_{a\in\mathbb{F}_q} f_{w_0U_{\alpha_2}(a)} + qf_{s_2s_1} + (q-1)\sum_{a\in\mathbb{F}_q} f_{w_0U_{\alpha_2}(a)} \\ &= qf_{s_2s_1} + q\sum_{a\in\mathbb{F}_q} f_{w_0U_{\alpha_2}(a)}. \end{aligned}$$

We will prove later that such a vector must generate the unique unipotent representation corresponding to the $[2,1]$ partition.

3.4.2 $GL_4(\mathbb{F}_q)$: The $\Gamma_{[2,2]}$ gGGr

This case illustrates some of the complexities which occur with the gGGr construction when one has a nonabelian subgroup $Z_L([\eta])$. As seen above, $Z_L([\eta])$ is isomorphic to $GL_2(\mathbb{F}_q)$, which has higher dimensional representations which support B-fixed vectors, complicating the study of the refined representations. In this section we compute the dimensions of the $\mathcal{H}_0$-modules $\left(\Gamma_{X,\rho}\right)^B$, but for brevity do not compute the explicit $\mathcal{H}_0$-module structure.

As mentioned before, the $[2,2]$ nilpotent orbit of $G = GL_4(\mathbb{F}_q)$ has $\tilde{X} = e_{\alpha_1+\alpha_2}(1) + e_{\alpha_2+\alpha_3}(1)$ as a representative. We have $\mathfrak{u}_1 = \mathfrak{u}_2 = \bigoplus_{\alpha\geq\alpha_2} e_\alpha$ and $Z_L([\eta])$ is $GL_2(\mathbb{F}_q)$, diagonally embedded in the $[2,2]$ block-diagonal matrices. The first order of business is to compute the dimension of $(\mathrm{ind}_{U_{1.5}}^{G}\eta_d)^B$. As a first approximation, we identify which Bruhat cells in G can support such a B-fixed vector. These will be exactly the Weyl group elements w such that $w(\alpha_1+\alpha_2)$ and $w(\alpha_2+\alpha_3)$ are not positive roots, which we can directly compute as $s_2s_1s_3w_0$, $s_1s_3w_0$, s_1w_0, s_2w_0, s_3w_0, and w_0.

As in the previous case, some of these cells will support more than one B-fixed vector in the corresponding gGGr. The Bruhat cell BwB will contain q^d $B\backslash G/U_{1.5}$ double cosets, where d is the number of positive root subgroups $U_\alpha \subseteq U_1/U_{1.5}$ such that $w(\alpha)$ is not a positive root. As before, if $BwU_{1.5}$ supports a B-fixed vector, and S is any set of positive roots such that for all $\beta \in S$, $w(\beta) \not< w(\alpha)$ for all α such that $U_\alpha \subseteq U_{1.5}$, then all double cosets of the form $Bw\prod_{\beta\in S} U_\beta(x_\beta)U_{1.5}$ support B-fixed vectors. These will all be distinct by the above, and they must support B-fixed vectors since $BwU_{1.5}$ supports a B-fixed vector and the condition on S implies $Bw\prod_{\beta\in S} U_\beta(x_\beta) \neq Bw\prod_{\beta\in S} U_\beta(x_\beta)u_{1.5}$ for any nonidentity $u_{1.5} \in U_{1.5}$. In particular, all $B\backslash G/U_{1.5}$ double cosets contained in the cells listed above will support B-fixed vectors except for possibly some from the s_2w_0 double coset. Ignoring this double coset, we have q^2+2q+2 B-fixed vectors. However, from Theorem 3.3 we know Γ_X^B should have dimension q^2+3q+2. In particular, exactly q of the q^2 double cosets associated to this Weyl group element must support B-fixed vectors.

In order identify these B-fixed vectors, we examine the relevant double coset at the 'matrix level.' Consider the double cosets with representatives $s_2w_0U_{\alpha_1}(x)U_{\alpha_3}(x)$. For such a double coset, define a function $f : Bs_2w_0U_{\alpha_1}(x)U_{\alpha_3}(x)U_{1.5} \to \mathbb{C}$ by

$$Bs_2w_0\begin{pmatrix}1 & x & k_1 & k_2\\ 0 & 1 & k_3 & k_4\\ 0 & 0 & 1 & x\\ 0 & 0 & 0 & 1\end{pmatrix} \mapsto \chi(k_1 - xk_3 + k_4).$$

Elements of $Bs_2w_0U_{\alpha_1}(x)U_{\alpha_3}(x)U_{1.5}$ have unique representations up to a choice of

$U_{\alpha_2}(b)$:

$$U_{\alpha_2}(b)s_2w_0\begin{pmatrix}1 & x & k_1 & k_2\\ 0 & 1 & k_3 & k_4\\ 0 & 0 & 1 & x\\ 0 & 0 & 0 & 1\end{pmatrix} = s_2w_0\begin{pmatrix}1 & x & k_1 & k_2\\ 0 & 1 & k_3+b & k_4+bx\\ 0 & 0 & 1 & x\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Then the function f is B-fixed since

$$f\left(BU_{\alpha_2}(b)s_2w_0\begin{pmatrix}1 & x & k_1 & k_2\\ 0 & 1 & k_3 & k_4\\ 0 & 0 & 1 & x\\ 0 & 0 & 0 & 1\end{pmatrix}\right) = f\left(s_2w_0\begin{pmatrix}1 & x & k_1 & k_2\\ 0 & 1 & k_3+b & k_4+bx\\ 0 & 0 & 1 & x\\ 0 & 0 & 0 & 1\end{pmatrix}\right)$$
$$= k_1 - x(k_3+b) + k_4 + xb = k_1 - xk_3 + k_4.$$

Now that we have identified all $B\backslash G/U_{1.5}$ double cosets which support B-fixed vectors, we can average on the right to find all $B\backslash G/Z_L(X)U_{1.5}$ double cosets which support B-fixed vectors in the refined decomposition of $\Gamma_{[2,2]}$. As in the $GL_3(\mathbb{F}_q)$ case above, considering $B\backslash G/Z_L([\eta])U_{1.5}$ double cosets allows us to consider a collection of cosets which can be described and enumerated independently of q. The double coset representatives are illustrated in the table below, where x and y are elements of $\mathbb{F}_q$.

$B\backslash G/Z_L([\eta])U_{1.5}$ Double Coset	$B\backslash G/U_{1.5}$ Double Cosets
$s_2s_1s_3w_0$	$s_2s_1s_3w_0$
	$s_2w_0U_{\alpha_1}(x)U_{\alpha_3}(x)$
$s_1s_3w_0$	$s_1s_3w_0$
	$w_0U_{\alpha_1}(x)U_{\alpha_3}(x)$
s_1w_0	$s_1w_0U_{\alpha_1}(x)$
	$s_3w_0U_{\alpha_3}(y)$
	$w_0U_{\alpha_1}(x)U_{\alpha_3}(y)$ $(x \neq y)$

It is noteworthy that the $B\backslash G/Z_L([\eta])U_{1.5}$ double coset which supports the most B-fixed vectors is not the one containing the long Weyl element, and thus many of the double cosets with representatives from the largest Bruhat cell do not land in the $B\backslash G/Z_L([\eta])U_{1.5}$ double coset represented by the long Weyl element. Each refined gGGr $\Gamma_{\tilde{X},\rho}$ which supports a B-fixed vector must have the representation ρ of $GL_2(\mathbb{F}_q)$ support a B-fixed vector. In particular,

$$(\Gamma_{\tilde{X}})^B = \left(\Gamma_{\rho_{triv},\tilde{X}}\right)^B \oplus q\left(\Gamma_{\rho_{St},\tilde{X}}\right)^B,$$

where ρ_{triv} denotes the trivial representation of $GL_2(\mathbb{F}_q)$ and ρ_{St} denotes the Steinburg representation.

By a dimension argument, the $s_2s_1s_3w_0$ double coset must support one B-fixed vector in both $\Gamma_{\rho_{triv},\tilde{X}}$ and $\Gamma_{\rho_{St},\tilde{X}}$. The remaining two double cosets therefore support $(q+1)^2$ B-fixed vectors together, meaning that at least one of these must support more than one B fixed vector in at least one of the refined gGGrs. To see where these 'extra' B-fixed vectors come from, it is useful to note that these double cosets together form the double coset for $B\backslash G/P$ represented by the long Weyl element (where P is the $[2,2]$ parabolic). In particular, for $\rho = \rho_{triv}$ or $\rho = \rho_{St}$, any representation of the Levi L which supports a B-fixed vector and occurs in $\pi = \mathrm{ind}_{GL_2(\mathbb{F}_q)}^{GL_2(\mathbb{F}_q)\times GL_2(\mathbb{F}_q)}\rho$ will contribute a B-fixed vector to the refined gGGr $\Gamma_{X,\rho}$ for one of these two double cosets. Since $\pi = \rho\otimes\rho_{reg}$, we find the following:

$$\begin{aligned}\dim\left(\Gamma_{\rho_{triv},\tilde{X}}\right)^B &= q+2\\ q\dim\left(\Gamma_{\rho_{St},\tilde{X}}\right)^B &= q^2+2q.\end{aligned}$$

We can also see these extra B-fixed vectors explicitly. Each $B\backslash G/U_{1.5}$ double coset in Bw_0P has representative of the form $w_0U_{\alpha_1}(x)U_{\alpha_2}(y)$. If f_ρ denotes the $B_{Z_L([\eta])}$-fixed vector of ρ, each B-fixed vector in the Bw_0P double coset of the refined gGGr $\Gamma_{\tilde{X},\rho}$ is of the form

$$f_\rho(bw_0U_{\alpha_1}(x)U_{\alpha_2}(y)u) = f_\rho(U_{\alpha_1}(x))\psi(y-x)\phi(u)$$

for some character ψ of $\mathbb{F}_q$. Since the dimensions of the refined gGGrs above depend on q, it is impractical to compute their matrices generally. We could compute the trace of certain elements in $\mathcal{H}_0$ acting on the above representations and appeal to character theory, but this can be a tricky computation which does not easily generalize. In the next section, we give a general description of which unipotent representations embed in (unrefined) gGGrs.

3.4.3 General Results for $GL_n(\mathbb{F}_q)$

Now we use the insights gleaned in the previous examples to state and prove some general results about the Hecke algebra structure of (refined) gGGrs for $GL_n(\mathbb{F}_q)$. As a corollary to one of our constructions, we will prove that the appearance of unipotent representations in refined gGGrs is predicted by the Springer correspondence in this case. In this section we will let Γ_λ denote the gGGr Γ_X where X is in the nilpotent orbit indexed by λ. Similarly, $\Gamma_{\lambda,\rho}$ will denote the refined gGGr using the representation ρ for $Z_L(X)$. First, we recall that from Theorem 3.3, the unipotent representation π_{ν^T} will occur in gGGr Γ_λ with multiplicity given by a Kostka-Foulkes polynomial. For this section, it is useful to think of this result in terms of Hecke algebras:

Theorem 3.6. *(Andrews and Thiem [1]) Let the notation be as above. Then*

$$(\Gamma_\lambda)^B \simeq \bigoplus_{\nu \geq \lambda} K_{\nu\lambda}(q)\tilde{\chi}_{\nu^T}.$$

In particular, $\dim \operatorname{Hom}_{\mathcal{H}_0}(\tilde{\chi}_{\lambda^T}, (\Gamma_\lambda)^B) = 1$.

This recalls a standard result about the representation theory of S_n, which states that the representations χ_ν occur in the λ-permutation module with multiplicity given by a Kostka number. Applying the Tits Deformation Theorem, we have the following result:

Proposition 3.7. *Let $\mathcal{H}_0$ be the Hecke algebra of S_n, and let $H_{0,\lambda}$ be the λ-*parabolic subalgebra *of $\mathcal{H}_0$.*[2] *Given a parabolic subalgebra $\mathcal{H}_{0,\lambda}$, let I_λ denote the one dimensional $\mathcal{H}_0$-module defined by $I_\lambda(T_w) = q^{\ell(w)}$ (the 'trivial' representation of $\mathcal{H}_{0,\lambda}$). Then*

$$\operatorname{ind}_{\mathcal{H}_{0,\lambda}}^{\mathcal{H}} I_\lambda \simeq \bigoplus_{\nu \geq \lambda} K_{\nu\lambda}\tilde{\chi}_\nu.$$

In particular, $\dim \operatorname{Hom}_{\mathcal{H}_0}(\tilde{\chi}_\lambda, \operatorname{ind}_{\mathcal{H}_{0,\lambda}}^{\mathcal{H}_0} I_\lambda) = 1.$

Note that in the '$\mathcal{H}_0$-module version' of the Andrews and Thiem result, $\tilde{\chi}_{\nu^T}$ embeds in $(\Gamma_\lambda)^B$ if and only if $\nu^T \leq \lambda$, where by the above result, $\tilde{\chi}_{\nu^T}$ embeds in $\operatorname{ind}_{\mathcal{H}_{0,\lambda^T}}^{\mathcal{H}} I_{\lambda^T}$ if and only if $\nu^T \geq \lambda^T$. In particular,

$$\dim \operatorname{Hom}_{\mathcal{H}_0}(\operatorname{ind}_{\mathcal{H}_{0,\lambda^T}}^{\mathcal{H}} I_{\lambda^T}, (\Gamma_\lambda)^B) = 1, \qquad (*)$$

where the unique $\mathcal{H}_0$-module in both $\operatorname{ind}_{\mathcal{H}_{0,\lambda^T}}^{\mathcal{H}_0} I_{\lambda^T}$ and $(\Gamma_\lambda)^B$ is $\tilde{\chi}_{\lambda^T}$. We leverage this fact to find specific generators for the distinguished $\mathcal{H}_0$-module $\tilde{\chi}_{\lambda^T}$ in $(\Gamma_\lambda)^B$. We recall from Proposition 2.6 that $U_{1.5}$ is the unipotent radical of a parabolic subgroup with Jordan block type given by a reordering of λ^T. Let $\mathcal{H}_{0,\tilde{\lambda}^T}$ denote the parabolic subalgebra of $\mathcal{H}_0$ given by the the 'opposite' block type, that is, the block type conjugated by w_0, and let $S_{\tilde{\lambda}^T}$ denote the corresponding parabolic subgroup of S_n. The following theorem gives us precise information about the copy of $\tilde{\chi}_{\lambda^T}$ in $(\Gamma_\lambda)^B$ (and by extension the copy of the unipotent representation π_{λ^T} in Γ_λ).

Theorem 3.8. *Choose a basis for $(\Gamma_\lambda)^B$ consisting of left B-invariant and right $U_{1.5}$-equivariant functions f_{wu} supported on the double coset $BwuU_{1.5}$ for $w \in W$ and $u \in U/U_{1.5}$, normalized so that $f_{wu}(wu) = 1$. For a given $w \in W$, let $\Phi_w := \{\alpha \in \Phi^+ \mid w(\alpha)$ is negative$\}$, and let $U_{\Phi_w} \subset GL_n(\mathbb{F}_q)$ denote the unipotent subgroup generated by the root spaces U_α for $\alpha \in \Phi_w$. Then the function*

$$f := \sum_{w \in S_{\tilde{\lambda}^T}} \sum_{u \in U_{\Phi_{ww_0}}} f_{ww_0u}$$

[2] *This is defined analogously to the permutation subgroup of S_n, i.e. taking the permutation representation on the standard basis $\{e_1, \ldots, e_n\}$, these are the elements with a λ block-diagonal structure.*

spans a copy of the $\mathcal{H}_{0,\tilde{\lambda}^T}$ module I_{λ^T} in $(\Gamma_\lambda)^B$. In particular, f generates the irreducible $\mathcal{H}_0$-module $\tilde{\chi}_{\lambda^T}$ in $(\Gamma_\lambda)^B$.

Proof. The second statement follows immediately from the first by $(*)$ and Frobenius reciprocity. To prove the first statement, we need to better understand the summands of f. Note that if U_α is a root subgroup in $U_{1.5}$, $w_0(\alpha)$ is a negative root which is not in the root space of the Levi $L_{\tilde{\lambda}^T}$. Then $(ww_0)(\alpha)$ is a negative root which is less than $(ww_0)(\beta)$ for all roots β of $L_{1.5}$. By an identical argument to that employed in the $[2,2]$ example, each summand f_{ww_0u} in the definition of f is nonzero and distinct summands are supported on distinct $B\backslash G/U_{1.5}$ double cosets.

To determine how the simple reflections $T_{s_\alpha} \in \mathcal{H}_{\tilde{\lambda}^T}$ act on f, we adapt the similar arguments from the discussion of $\Gamma_{[2,1]}$ for $GL_3(\mathbb{F}_q)$. Let s_α be a simple reflection in $S_{\tilde{\lambda}^T}$. If $\ell(s_\alpha ww_0) > \ell(ww_0)$, $w_0w^{-1}(\alpha)$ is positive, so

$$T_{s_\alpha} * f_{ww_0u} = \sum_{x\in\mathbb{F}_q} f_{s_\alpha ww_0 U_{w_0w^{-1}(\alpha)}(x)u}.$$

Similarly, if $\ell(s_\alpha ww_0) < \ell(ww_0)$, $w_0w^{-1}(-\alpha)$ is positive, so

$$T_{s_\alpha * f_{ww_0u}} = qf_{s_\alpha ww_0u} + \sum_{x\in\mathbb{F}_q^\times} f_{ww_0U_{w_0w^{-1}(-\alpha)}(x)u}.$$

These observations immediately show that each summand f_{ww_0u} in the definition of f is sent to a linear combination of functions which are also summands in the definition of f. Consider one such summand f_{ww_0u}. If $\ell(s_\alpha ww_0) > \ell(ww_0)$, the above equations imply the only summand of f contributing to the coefficient of f_{ww_0u} under the T_{s_α} action will be $f_{s_\alpha ww_0u}$. The coefficient of f_{ww_0u} in $T_{s_\alpha} * f_{ww_0u}$ is q by the above. Similarly, if $\ell(s_\alpha ww_0) < \ell(ww_0)$, the only summands of f contributing to the coefficient of f_{ww_0u} will be the $q-1$ summands of the form $f_{s_\alpha ww_0U_{w_0w^{-1}s_\alpha(\alpha)}(x)u}$ for $x \neq 0$ and the function f_{ww_0u}. It follows that $T_{s_\alpha} * f = qf$ for each $s_\alpha \in S_{\tilde{\lambda}^T}$. The result follows since $\mathcal{H}_0$ is generated by the elements T_{s_α}.

$\square$

Now that we have identified a generator of π_{λ^T} in Γ_λ, to verify Kawanaka's conjecture we simply need to place π in the decomposition

$$\Gamma_\lambda = \bigoplus_{\rho \in Z_L([\eta])} (\dim \rho) \Gamma_{\rho,\lambda}$$

from Proposition 2.4. By construction, f is identically one on $w_0 L$, so f has a nonzero image under the projection to $\Gamma_{\rho_{triv},\lambda}$ in this decomposition. Since f generates π_{λ^T}, we have the following solution to our main problem for $GL_n(\mathbb{F}_q)$:

Corollary 3.9. *The unipotent representation π_{λ^T} in $G = GL_n(\mathbb{F}_q)$ embeds uniquely in the refined gGGr $\Gamma_{\lambda,\rho_{triv}}$.*

As mentioned above, ρ_{triv} can be interpreted as a representation of the G-equivariant fundamental group for the nilpotent orbit given by λ, and π_{λ^T} has an $\mathcal{H}_0$-module which is a q-deformation of the Specht module χ_λ. If X_λ is the nilpotent element constructed by Dynkin-Kostant theory, the bijection

$$\chi_{\lambda^T} \leftrightarrow (\lambda, \rho_{triv})$$

is exactly the one given by the Springer correspondence for $G = GL_n(\mathbb{F}_q)$.

Chapter 4

Models, Functionals, and Hecke Algebras in the p-Adic Setting

We now turn our attention to studying unique models and unique functionals in the p-adic setting. We begin by establishing standard notation which will be in use for the remainder of this document. From this point on, we will have $G = G(\mathbb{F})$ for a p-adic field $\mathbb{F}$. Let $\mathfrak{o}$ denote the ring of integers of $\mathbb{F}$, ϖ denote a uniformizer in $\mathfrak{o}$, $\mathfrak{p} := \varpi\mathfrak{o}$ denote the maximal prime ideal in $\mathfrak{o}$, and $q := |\mathfrak{o}/\mathfrak{p}|$ be the residue characteristic (assumed large as in the finite case). Let Φ be the root system for G, with positive roots Φ^+ and negative roots Φ^-. Let $B = TU$ be the Borel subgroup of G with respect to this choice of positive roots, where here T is a maximal split torus and U is the unipotent radical of B. Let W be the Weyl group for G and S be the simple reflections in W. Let X_* be the cocharacter lattice of T. Let I be the Iwahori subgroup of G, that is, the inverse image of the Borel subgroup under the projection $G(\mathfrak{o}) \to G(\mathfrak{o}/\mathfrak{p})$. For the remainder of this document, let ind denote compact induction, and if $P = MN$ is a parabolic subgroup of G and ρ is a representation of M, let $i_P^G\rho$ denote compactly supported parabolic induction of ρ to G.

4.1 The Iwahori-Hecke Algebra

The Iwahori Hecke algebra of G is the the convolution algebra $\mathcal{H} = C_c^\infty(G//I, *)$. Our discussion of the Iwahori Hecke algebra is heavily informed by the exposition in Haines, Kottwitz, and Prasad [25]. By the Iwasawa decomposition, $G = ITK$ where $K := G(\mathfrak{o})$ is the maximal compact subgroup of G. Lifting the Bruhat decomposition of $G(\mathbb{F}_q)$ to K, we have $K = IWI$, so $G = ITIWI = I(T/T(\mathfrak{o}))WI$. $T/T(\mathfrak{o})$ has coset representatives of the form $\varpi^\mu := \mu(\varpi)$ for $\mu \in X_*$. Using the bijection $T/T(\mathfrak{o}) \leftrightarrow X_*$, we see that $\mathcal{H}$ has a basis of functions $T_w := \mathbb{1}_{IwI}$ where the coset representative w is in the extended affine Weyl group $\widetilde{W} = W \ltimes X_*$. Let $R = \mathbb{C}[X_*] = \mathbb{C}[T/T(\mathfrak{o})]$. It follows that there is an isomorphism of complex vector spaces

$$\mathcal{H} \simeq \mathcal{H}_0 \otimes R \simeq \mathbb{C}[\widetilde{W}].$$

Thus the Iwahori-Hecke algebra is generated by the finite Hecke algebra generators together with generators indexed by cocharacters of the torus. We write the Hecke algebra generator corresponding to the cocharacter μ as π^μ corresponding to the double coset $I\mu(\pi)I$ when μ is dominant. Many authors, including Lusztig, write π^μ as $\Theta(\mu)$. The *Bernstein presentation* of the Iwahori-Hecke algebra is $\mathcal{H} = \mathcal{H}_0 \ltimes R$, where $\mathcal{H}_0$ has the usual Hecke algebra relations

$$T_i^2 = (q-1)T_i + q \qquad \text{(quadratic relations)}$$
$$\underbrace{T_iT_jT_i}_{m_{i,j}\ \text{factors}} \ldots = \underbrace{T_jT_iT_j\ldots}_{m_{i,j}\ \text{factors}} \qquad \text{(braid relations),}$$

R is commutative, and commutation between T_i and π^μ is given by the *Bernstein relation*

$$T_i\pi^\mu = \pi^{s_i(\mu)} + (1-q)\frac{\pi^{s_i(\mu)} - \pi^\mu}{1 - \pi^{-\alpha^\vee}}.$$

As in the case of the finite Hecke algebra above, if the residue characteristic q is viewed as a parameter, the Iwahori-Hecke algebra is a deformation of the group algebra $\mathbb{C}[\widetilde{W}]$.

Hecke algebras play an important role in the representation theory of reductive groups over p-adic fields. In general, if K is a compact, open subgroup of G and π is a smooth representation of G, the Hecke algebra $\mathcal{H}_K := C_c^\infty(G//K, *)$ acts on the K-fixed vectors π^K in π by

$$\pi(h) \cdot v := \int_G h(g)\pi(g) \cdot v \, dg \qquad v \in \pi^K, h \in \mathcal{H}_K.$$

This action establishes a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of irreducible,} \\ \text{smooth representations with } K\text{-fixed vectors} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{simple} \\ \mathcal{H}\text{-modules} \end{array} \right\}$$

The following version of the Borel-Matsumoto theorem, as stated in [15], is particularly useful for our purposes.

Theorem 4.1. *(Borel-Matsumoto, Casselman) There is a category equivalence between irreducible, admissible representations which have a non-zero I-fixed vector and finite dimensional representations of $\mathcal{H}$. Irreducible representations which have a non-zero I-fixed vector are exactly subquotients of unramified principal series representations $i_B^G\chi$ for an unramified character χ of T.*

As mentioned above, the unramified principal series representations play an important role in number theory, as for any global automorphic representation $\pi_{\mathbb{A}} = \bigotimes_p \pi_p$, almost all of the local factors π_p are unramified principal series representations. We further expand the connections between unramified principal series representations and the Iwahori-Hecke algebra in the next section.

4.2 The Universal Unramified Principal Series

Here we discuss the close connection between the unramified principal series and the Iwahori-Hecke algebra. In this section, we review the theory of the universal unramified principal series as described by Haines, Kottwitz, and Prasad [25]. This structure allows us to study the full class of unramified principal series representations, illuminating additional algebraic structure. Let $\chi_{univ} : T/T(\mathfrak{o}) \to R$ denote the "universal" character of the torus defined by $\pi^\mu \mapsto \pi^\mu$, and let $i_B^G \chi_{univ}^{-1}$ be the universal unramified principal series.

There are isomorphisms $i_B^G \chi_{univ^{-1}} \simeq i_B^G R \simeq C_c^\infty(T(\mathfrak{o})U\backslash G)$. While the first isomorphism is essentially a notational choice, the second isomorphism is more complex. For $\varphi \in C_c^\infty(T(\mathfrak{o})U\backslash G)$, the corresponding element of $i_B^G R$ is

$$\phi(g) := \sum_{a \in T/T(\mathfrak{o})} \delta_B(a)^{-1/2} \varphi(ag) \cdot a,$$

where δ_B denotes the Haar measure on B. While we will not use this explicit isomorphism in our later work, we draw attention to it here to emphasize that the isomorphism we describe later in this document are not trivial and may be difficult to compute explicitly.

This construction is universal in the following way. Any linear character $\chi : T/T(\mathfrak{o}) \to \mathbb{C}^\times$ determines a character $\mathbb{R} \to \mathbb{C}$, and we have an $\mathcal{H}$-module isomorphism

$$\mathbb{C} \otimes_R (i_B^G R)^I = (i_B^G \chi^{-1})^I.$$

We define $\mathcal{M} := C_c^\infty(T(\mathfrak{o})U\backslash G/I)$, the Iwahori-fixed vectors in the universal unramified principal series. The module $\mathcal{M}$ is spanned by vectors v_x supported on $T(\mathfrak{o})UxI$ for $x \in \widetilde{W}$. There is a left R-action on $\mathcal{M}$ by $\pi^\mu \cdot v_x = q^{-\langle \rho, \mu \rangle} v_{\pi^\mu \cdot x}$ which commutes with the right $\mathcal{H}$ action on $\mathcal{M}$ by convolution, giving $\mathcal{M}$ the structure of an (R, H)-bimodule. There is an isomorphism $\mathcal{M} \simeq \mathcal{H}$ of right $\mathcal{H}$-modules. This final fact is particularly important to us: for any representation V of G, we have right $\mathcal{H}$-module isomorphisms

$$V^I \simeq \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}, V) \simeq \mathrm{Hom}_{\mathcal{H}}(\mathcal{M}, V),$$

Thus, the connection of the universal unramified principal series to model spaces (which are, among other things, G-representations) is both clear and unsurprising. Remarkably, many unique models can be associated to a *left* $\mathcal{H}$-module structure, which we now describe.

We can make $\mathcal{M}$ into a left $\mathcal{H}$-module by defining $h'(v_1h) := v_1h'h$. The map $v_1h \mapsto v_1h'h$ is an endomorphism of $\mathcal{M}$ as a right $\mathcal{H}$-module, and we have an isomorphism $\mathcal{H} \simeq \mathrm{End}_{\mathcal{H}}(\mathcal{M})$. We let $\phi_1 \in i_B^G R$ correspond to v_1 under the isomorphism $\mathcal{M} \simeq (i_B^G R)^I$, and observe that this can be used to directly define a left $\mathcal{H}$-action on $(i_B^G R)^I$. This left action is closely related to the "universal" analogues of usual intertwining operators for principal series representations. For each $w \in W$, there is a (normalized) intertwining operator $\mathcal{A}_w$ on $i_B^G R$ given by

$$\mathcal{A}_w : \varphi \mapsto \left[\prod_{\alpha \in \Phi_w^+} (1 - \pi^{\alpha^\vee})\right] \int_{U \cap w\overline{U}w^{-1}} \varphi(w^{-1}ug)\, du,$$

where $\Phi_w^+ := \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\}$. For a simple reflection $s \in S$, the intertwiner $\mathcal{A}_s$ is associated to the endomorphism $T_s \in \mathcal{H} \simeq \mathrm{End}_G(\mathcal{M})$ by

$$q^{-1}(1 - \pi^{\alpha^\vee})T_{s_\alpha} \cdot \phi = \mathcal{A}_{s_\alpha}\phi + (q^{-1} - 1)\pi^{\alpha^\vee}.$$

4.3 $\mathcal{H}$-Modules Associated to Unique Functionals

In this section we explain formalism allowing us to discuss unique functionals on the universal unramified principal series. We define an *R-valued functional* to be an R-linear map

$$\mathcal{L} : i_B^G R \to R.$$

If H is a closed subgroup of G and ψ is a linear character of H, we often consider *H, ψ equivariant functionals* $\mathcal{L}$ (i.e. those satisfying $\mathcal{L}(h\phi) = \psi(h)\mathcal{L}(\phi)$ for $h \in H$). In the case that $H = U$ and ψ is a nondegenerate character, there is a unique (up to scalars in R) U, ψ-equivariant functional called the *R-valued Whittaker functional*. As in the case of the usual Whittaker functional, uniqueness in this setting imposes a quite rigid structure. Before we discuss this further, we must first provide a sufficiently general definition of uniqueness in the context of R-valued functionals.

Definition 4.2. *Let P be any property associated to R-valued functionals on the universal unramified principal series. If both*

(a) P is preserved by automorphisms of $i_B^G R$, and

(b) any two R-valued functionals with property P differ by a scalar in R,

we say any R-valued functional which satisfies property P is unique *(with respect to P).*

The property P in the above definition is often an equivariance property as discussed above, but we use this broad language to allow for R-valued analogues of other p-adic functionals, such as those arising from spherical varieties or more complex model spaces. If $\mathcal{L}$ is a unique R-valued functional, then Schur's lemma guarantees that for each $w \in W$, there is a functional equation

$$\mathcal{L} \circ \mathcal{A}_w(\phi) = b_w(\mathcal{L})\mathcal{L}(\phi)^w$$

for some scalar $b_w(\mathcal{L})$. In particular, the left endomorphism action of $\mathcal{H}$ on $i_B^G R$ transfers to the image space in the functional. We make this notion precise with the following Lemma.

Lemma 4.3. *Let $\mathcal{L} : i_B^G(\chi_{\text{univ}}^{-1}) \to V$ be a unique functional such that $V^I \hookrightarrow R$ as a right R-module. Then the map*

$$\mathcal{L} : \begin{array}{ccc} i_B^G(\chi_{\text{univ}}^{-1})^I & \longrightarrow & V^I \\ \phi & \longmapsto & \mathcal{L}(\phi) \end{array}$$

induces a left $\mathcal{H}$-module structure on V^I in which R acts by translation on the image of $\mathcal{H}$ in V^I and T_s acts by

$$T_s \cdot f := q(1 - \pi^{\alpha^\vee})^{-1} \left(b_s(\mathcal{L}) f^s + (q^{-1} - 1)\pi^{\alpha^\vee} f \right)$$

for a constant $b_s(\mathcal{L}) \in R$. In particular, each unique, R-valued functional $\mathcal{L} : \mathcal{M} \to R$ is completely determined by $\mathcal{L}(\phi_1)$ and $b_s(\mathcal{L})$ for $s \in S$.

Proof. That R inside $\mathcal{H}$ acts by translation just follows from the fact that, for any $\phi \in \mathcal{M}$, $\pi^\mu \cdot \phi$ is given by writing $\phi = \phi_1 * h$ for some $h \in \mathcal{H}$, for which the left action by π^μ is

$$\pi^\mu \cdot \phi = \phi_1 * \pi^\mu h = (\pi^\mu \phi_1) * h = \pi^\mu(\phi_1 * h) = \pi^\mu \phi.$$

Thus, we may extract π^μ multiplying ϕ from each R-linear functional $\mathcal{L}_i$.

Moreover, uniqueness of the model induces an action of T_s for s a simple reflection on V by exploiting the relation between T_s and $\mathcal{A}_s$ for the simple reflection $s = s_\alpha$. Indeed, using the relationship between T_s and $\mathcal{A}_s$, we may define the action $T_s \cdot f$ on the image of $\mathcal{L}$ in V by

$$T_s \cdot f := q(1 - \pi^{\alpha^\vee})^{-1} \left(b_s(\mathcal{L}) f^s + (q^{-1} - 1)\pi^{\alpha^\vee} f \right)$$

where the constant $b_s(\mathcal{L})$ occurs in the the functional equation $b_s(\mathcal{L})\mathcal{L}(\phi)^s = \mathcal{L} \circ \mathcal{A}_s$. Then by construction $T_s \cdot \mathcal{L}(\phi)$ agrees with $\mathcal{L}(T_s \cdot \phi)$, so it automatically satisfies relations in the Hecke algebra. This gives the left $\mathcal{H}$-module structure on the image of $\mathcal{L}$ in V^I.

The final assertion that each unique R-valued functional $\mathcal{L} : \mathcal{M} \to R$ is determined by $\mathcal{L}(\phi_1)$ and the constants $b_s(\mathcal{L})$ follows from the fact that $\mathcal{M} = \mathcal{H}\phi_1$ and the fact that $\mathcal{L}$ is a left $\mathcal{H}$-module morphism with structure constants depending only on the elements $b_s(\mathcal{L}) \in R$. □

4.4 Connections Between Models and Functionals

We now have two $\mathcal{H}$-modules emanating from this theory - the right $\mathcal{H}$-module of Iwahori-fixed vectors in a model space, and the left $\mathcal{H}$-module obtained by the action on the image of a unique R-valued functional on $\mathcal{M}$. From Chan and Savin's work [12, 13], we know that the spherical model, the Whittaker model, and a split Bessel model for SO_{2n+1} have Iwahori component isomorphic to $\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}}\,\varepsilon$ as a right $\mathcal{H}$-module (where again, ε is a linear character of $\mathcal{H}_0$). Comparing these results those of Brubaker, Bump, and Friedberg [4], for both the spherical model, the Whittaker model, and the Bessel model for odd orthogonal groups, the *left* $\mathcal{H}$-module obtained by acting on the image of the functional is also of the form $\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}}\,\varepsilon$ (where here the induction is as a right $\mathcal{H}$-module). Moreover, the character ε from the left $\mathcal{H}$-module on the functional side matches the character ε from the right $\mathcal{H}$-module on the model side in all cases. It is not immediately clear why this should be the case. Indeed, there is no natural way to put a right $\mathcal{H}$-module structure on the image of a functional. This section follows an idea of Savin in the Whittaker case [40] to show that, in a large number of cases, the left $\mathcal{H}$-action on the image of a functional must be "the same as" the right $\mathcal{H}$-action on a corresponding model space.

We begin by providing a general framework in which the above examples fit. Let H be a closed subgroup of G and let ψ be a smooth character of H. Let $\mathcal{J}_{H,\psi}$ denote the "functional of H coinvariants"

$$\mathcal{J}_{H,\psi}(V) := V/\left\langle h\cdot v - \psi(h)v \mid v \in V, h \in H\right\rangle$$

for a representation V of G. Many of the nice properties from the Whittaker model case follow from the exactness of the (twisted) Jacquet functor. It turns out that we have exactness whenever H is exhausted by its compact open subgroups (i.e. for each $h \in H$, there is a compact open subgroup H_c of H which contains h). We are particularly interested in the case where H is a subgroup of a parabolic subgroup $P = LU$ with unipotent radical U and Levi subgroup L. Every subgroup of $L(\mathfrak{o})U$ is exhausted by its compact open subgroups. [1] The following Lemma is explicated in many places, including notes from Conrad's seminar [17].

Lemma 4.4. *Assume H is exhausted by its compact open subgroups. For a smooth representation V of G, define $V(H,\psi) := \langle \pi(h)v - \psi(h)v \rangle$. Then $v \in V(H,\psi)$ if and only if there exists a compact subgroup $H_c \subset H$ such that*

$$e_{H_c,\psi}(v) := \int_{H_c} \psi(h^{-1})\pi(h) \cdot v \, dH_c = 0.$$

Proof. Let $v = \sum_{i=1}^{k} \pi(h_i)v_i - \psi(h_i)v_i \in V(H,\psi)$. By assumption, there is a compact open subgroup $H_c \subseteq H$ containing $\{h_i\}_{i=1}^{k}$. Then for a given i, we have for any choice of right Haar measure on H_c,

$$\int_{H_c} \psi(h^{-1})\pi(h)\pi(h_i)v_i \, dH_c = \int_{H_c} \psi(h_i h^{-1})\pi(h)v_i \, dH_c = \int_{H_c} \psi(h^{-1})\pi(h)\psi(h_i)v_i \, dH_c,$$

using the change of variables $h \mapsto hh_i^{-1}$. In particular, $e_{H_c,\psi}(\pi(h_i)v_i) = e_{H_c,\psi}(\psi(h_i)v_i)$ for all i, so $e_{H_c,\psi}(v) = 0$.

Now assume $e_{H_c,\psi}(v) = 0$. Pick $H_c' \subseteq \ker\psi \cap H_c$ small enough to stabilize v under the π action. Let $h_1, \ldots, h_r$ be coset representatives of H_c' in H_c. Then

$$\begin{aligned} e_{H_c,\psi}(v) &= \int_{H_c} \psi(h^{-1})\pi(h)v \, d_{H_c} \\ &= \tfrac{1}{r} \sum_{i=1}^{r} \psi(h_i)\pi(h_i)v, \end{aligned}$$

[1] To see this, let $U = U_1 \supset U_2 \supset \cdots \supset U_r = 1$ denote the (lower) central series of U (so $U_i := [U, U_{i-1}]$). Let $N_1, \ldots, N_{r-1}$ denote the corresponding complementary subsets in U, that is, $[N_i, N_i] \subseteq U_{i+1}$ and $N_i \cap U_{i+1} = \emptyset$. Then for any $n \in \mathbb{Z}$, U has a compact, open subgroup of the form $N_1(\mathfrak{p}^n)N_2(\mathfrak{p}^{n-1})\ldots N_{r-1}(\mathfrak{p}^{n-r+1})$. The result follows since the $L(\mathfrak{o})$-action preserves spheres in U.

which is zero by assumption. Then

$$v = v - e_{H_c,\psi}(v) = v - \tfrac{1}{r}\sum_{i=1}^{r}\psi(h_i)^{-1}\pi(h_i)v = \tfrac{1}{r}\sum_{i=1}^{r} v - \psi(h_i)^{-1}\pi(h_i)v.$$

Letting $v_i := \frac{\psi(h_i)^{-1}}{r}v$, we have

$$v = \sum_{i=1}^{r}\pi(h_i)v_i - \psi(h_i)v_i \in V(H,\psi),$$

as desired. □

Proposition 4.5. *Assume H is exhausted by its compact open subgroups. Then $\mathcal{J}_{H,\psi}$ is an exact functor from $H - Mod$ to $H/\ker\psi - Mod$.*

Proof. Here we follow the argument of Bump's [5]. We first show that the functor $V \mapsto V(H,\psi)$ is exact. Consider a short exact sequence of H-modules

$$0 \longrightarrow V' \xrightarrow{i} V \xrightarrow{p} V'' \longrightarrow 0.$$

It is sufficient to show that the induced sequence

$$0 \longrightarrow V'(H,\psi) \xrightarrow{i_{H,\psi}} V(H,\psi) \xrightarrow{p_{H,\psi}} V''(H,\psi) \longrightarrow 0.$$

is exact. Viewing V' as a submodule of V, we certainly have $V'(H,\psi) \subseteq V(H,\psi)$, so the induced map $i_{H,\psi}$ is indeed injective. To see that $p_{H,\psi}$ is surjective, let $h \cdot v'' - \psi(h)v'' \in V''(H,\psi)$. By the surjectivity of p, there is some $v \in V$ with $p(v) = v''$. Then $p_{H,\psi}(h \cdot v - \psi(h)v) = (h \cdot v'' - \psi(h)v'')$. Finally, to see exactness at $V(H,\psi)$, consider $v \in V$ with $p(v) = 0$. Then $v \in V' \cap V(H,\psi)$ by exactness of the original sequence. By Lemma 4.4, since $v \in V(H,\psi)$, there is a compact subgroup $H_c \subset H$ such that

$$\int_{H_c}\psi(h^{-1})h \cdot v\, dH_c = 0.$$

Since $v \in V'$, the above and Lemma 4.4 says $v \in V'(H,\psi)$, so $V'(H,\psi) = V' \cap V(H,\psi)$.

Exactness follows since $V' = \ker p$.

Define inclusion maps $j : V(H,\psi) \hookrightarrow V$, $j' : V'(H,\psi) \hookrightarrow V'$, and $j'' : V''(H,\psi) \hookrightarrow V''$. We have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & V'(H,\psi) \arrow[r, "i_{H,\psi}"] \arrow[d, "j'"] & V(H,\psi) \arrow[r, "p_{H,\psi}"] \arrow[d, "j"] & V''(H,\psi) \arrow[r] \arrow[d, "j''"] & 0 \\
0 \arrow[r] & V' \arrow[r, "i"] & V \arrow[r, "p"] & V'' \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma, there is an exact sequence $\ker(j'') \to \operatorname{coker}(j') \to \operatorname{coker}(j) \to \operatorname{coker}(j'') \to 0$. Since j'' is an inclusion map, $\ker(j'') = 0$. Then we have a short exact sequence

$$0 \longrightarrow \mathcal{J}_{H,\psi}(V') \longrightarrow \mathcal{J}_{H,\psi}(V) \longrightarrow \mathcal{J}_{H,\psi}(V'') \longrightarrow 0$$

obtained by identifying $\mathcal{J}_{H,\psi}(V')$ with $\operatorname{coker}(j')$, $\mathcal{J}_{H,\psi}(V)$ with $\operatorname{coker}(j)$, and $\mathcal{J}_{H,\psi}(V'')$ with $\operatorname{coker}(j'')$. It follows that $\mathcal{J}_{H,\psi}$ is exact. □

We now establish the effect of the functors $\mathcal{J}_{H,\psi}$ on induced representations.

Lemma 4.6. *Let L be a closed subgroup of G and let θ be a linear character of L. There is an isomorphism of $N_G(\psi)/\ker\psi$-modules*

$$\mathcal{J}_{H,\psi}(\operatorname{ind}_L^G \theta) \simeq F_{H,\psi} := \left\{ f \in C^\infty(G) \,\middle|\, \begin{array}{c} f(lgh) = \theta(l)f(g)\psi(h) \\ \text{and } f \text{ is compact mod } L \text{ on the left and } H \text{ on the right} \end{array} \right\}$$

and $\mathcal{J}_{H,\psi}$ is given by $f \mapsto e_{H,\psi}(f) := \int_H \psi(h^{-1})\pi(h)f\,dh$.

Proof. By Lemma 4.4,

$$\ker \mathcal{J}_{H,\psi} = \bigcup_{\substack{H_c \subset H \\ \text{compact}}} \ker e_{H_c,\psi}.$$

Moreover, the first part of the proof of Lemma 4.4 shows that if $H_c' \subset H_c$ and $e_{H_c',\psi}(f) = 0$, then $e_{H_c,\psi}(f) = 0$. It follows that $f \in \ker \mathcal{J}_{H,\psi}$ if and only if $e_{H_c,\psi}(f) = 0$ for sufficiently large H_c. In the case that f is compactly supported, the integral

$$e_{H,\psi}(f) := \int_H \psi(h^{-1})\pi(h)f \, dh$$

converges. Then in this case, $\ker \mathcal{J}_{H,\psi} = \ker e_{H,\psi}$, so here $\mathcal{J}_{H,\psi}$ is realized by an integral. Moreover, for $f \in \operatorname{ind}_L^G \theta$, the function $e_{H,\psi}(f)$ is compact mod L on the left and mod H on the right, and as in the proof of Lemma 4.4, $e_{H,\psi}(f)$ is right H-equivariant by ψ. Thus $e_{H,\psi}$ maps $\operatorname{ind}_L^G \theta \to F_{H,\psi}$.

We now show that $e_{H,\psi}$ is surjective. We note that $F_{H,\psi}$ is spanned by functions f_x supported on $LxKH$, where K is a compact open subgroup of G satisfying $K \cap H \subset \ker \psi$, $x \in LK \backslash G/H$, and $f_x(lkxh) = \theta(l)\psi(h)f(x)$. Let f_x be such a function. Then $f_x|_{LK} \in \operatorname{ind}_L^G \theta$, and $e_{H,\psi}(f_{LK})$ is also supported on $LKxH$. Moreover, for any $l \in L$, $k \in K$, and $h \in H$,

$$\begin{aligned} e_{H,\psi}((1/\mu(K \cap H))f|_{LK})(lxkh) &= \frac{1}{\mu(K \cap H)} \int_H \psi(y^{-1})\pi(y)f|_{LK}(lxkh)\, dx \\ &= \frac{1}{\mu(K \cap H)} \int_H \psi(y^{-1}h)f|_{LK}(lxky)\, dx \\ &= \theta(l)f(x)\psi(h) = f(lxkh). \end{aligned}$$

The result follows. □

The most common example of this kind of equivariant functor is, of course, the Jacquet functor. For any parabolic subgroup $P = LU$ of G, the Jacquet functor is $\mathcal{J}_{U,\mathbb{1}}$, and is adjoint to parabolic induction from the Levi factor L. The real contribution of the above arguments is that we can enlarge the subgroup U to UL_0 for a compact subgroup L_0 of L, and we can replace the trivial character with a different linear character. The Jacquet functor plays particularly nicely with the Iwahori-Hecke algebra, as illustrated by the following result.

Theorem 4.7. *(Borel, Casselman, Matsumoto, and Bernsetin [2, 7]) Let V be a smooth representation of G. There is an isomorphism of $R \simeq \mathbb{C}[T/T(\mathfrak{o})]$-modules*

$$V^I \cong V_U^{I \cap T}$$

defined via the natural map $V \to V_U$.

The following Lemma from [40] is crucial to our later arguments.

Lemma 4.8. *There are isomorphisms of left G-modules $i_B^G R \simeq C_c^\infty(UT(\mathfrak{o})\backslash G) \simeq C_c^\infty(I\backslash G)$ where the G action is given by translation on the right.*

Proof. The first isomorphism was given in our earlier discussion and is shown in [25]. By Theorem 4.7, for any smooth representation π of G, there is a vector space isomorphism $\pi^I \to \pi_U^{T(\mathfrak{o})}$. Thus, considering $C_c^\infty(G)$ as a *right* G-module by *left* translation and applying Lemma 4.6, we have a vector space isomorphism

$$\Phi : C_c^\infty(I\backslash G) \to C_c^\infty(UT(\mathfrak{o})\backslash G).$$

As the map Φ is defined by a quotient on the left, it must commute with the G-action by right translations. It follows that Φ is an isomorphism of right G-modules. □

With our existing setup, we can give a particularly nice proof that the endomorphism ring of the universal unramified principal series is indeed the Iwahori-Hecke algebra. This provides an alternative description of the left $\mathcal{H}$ action on the unramified principal series representation. We emphasize that in many ways, the "hard work" is being done by the chain of isomorphisms in Lemma 4.8. It would be interesting to compute these isomorphisms explicitly on a chosen basis for the universal unramified principal series.

Proposition 4.9. *There is an isomorphism of algebras $\mathcal{H} \simeq \mathrm{End}_G(i_B^G R)$ which establishes a left $\mathcal{H}$-action on $i_B^G R$.*

Proof. We use the description $i_B^G R \simeq C_c^\infty(I\backslash G) = \mathrm{ind}_I^G \rho_{triv}$ from Lemma 4.8. Then by p-adic Mackey theory, $\mathrm{End}_G(\mathrm{ind}_I^G \rho_{triv}) = C_c^\infty(I\backslash G/I)$, and is generated by convolution integrals of the form

$$I_w(f)(x) := \int_G \mathbb{1}_{IwI}(xg^{-1})f(g)\,dg = \int_G \mathbb{1}_{IwI}(g)f(g^{-1}x)\,dg \qquad w \in \widetilde{W}$$

where for $X \subset G$, $\mathbb{1}_X$ denotes the characteristic function of X. Composition of the operators I_{w_1} and I_{w_2} is

$$\begin{aligned} I_{w_1} \circ I_{w_2}(f)(x) &= \int_G \mathbb{1}_{Iw_1I}(g_1) \int_G \mathbb{1}_{Iw_2I}(g_1^{-1}xg_2^{-1})f(g_2)\,dg_2\,dg_1 \\ &= \int_G \left(\int_G \mathbb{1}_{Iw_1I}(g_1)\mathbb{1}_{Iw_2I}(g_1^{-1}g_2)\,dg_1 \right) f(g_2^{-1}x)\,dg_2 \qquad g_2 \mapsto g_2^{-1}x \\ &= I_{w_1} * I_{w_2}(f)(x). \end{aligned}$$

Thus, composition of the operators I_{w_1} and I_{w_2} corresponds to the convolution operation on $C_c^\infty(G//I)$. The result follows. □

We have now established all of the necessary machinery to show some explicit connections between models and functionals. As noted above, many well known unique models and unique functionals are associated to $\mathcal{H}$-modules of the form $\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$ for a linear character ε of $\mathcal{H}$. These arguments follow ideas from a correspondence with Savin [40].

Theorem 4.10. *Let H be a closed subgroup of G which is exhausted by its compact open subgroups, and let ψ be any linear character of H. Then the left $\mathcal{H}$-action on $\mathcal{J}_{H,\psi}(i_B^G R)$ is the same as the right $\mathcal{H}$-action on $(\mathrm{ind}_H^G \psi)^I$.*[2]

Proof. By Lemma 4.8, there is a left G-module by right translation isomorphism $i_B^G R \simeq C_c^\infty(I\backslash G)$. By Mackey theory, $\mathrm{End}_G(I\backslash G) \simeq \mathcal{H}$. Thus, the isomorphism $i_B^G R \simeq C_c^\infty(I\backslash G)$ identifies the left $\mathcal{H}$-action on $i_B^G R$ by endomorphisms with the left $\mathcal{H}$-action on $C_c^\infty(I\backslash G)$ by convolution. Since $\mathcal{J}_{H,\psi}$ is exact, $\mathcal{J}_{H,\psi}$ gives a homomorphism $\mathrm{End}_G(C_c^\infty(I\backslash G)) \to \mathrm{End}_{H/\ker\psi}(\mathcal{J}_{H,\psi} C_c^\infty(I\backslash G))$. Moreover, by Lemma 4.6, $\mathcal{J}_{H,\psi}$ is defined by an integral on

[2] *By "the same as", I mean there is a canonical bijection $\mathcal{F} : \mathcal{J}_{H,\psi}(i_B^G R) \to (\mathrm{ind}_H^G \psi)^I$, such that, if $h_1, h_2 \in \mathcal{H}$ and $v \in \mathcal{J}_{H,\psi}(i_B^G R)$, $\mathcal{F}(h_1 h_2(v)) = \mathcal{F}(v) h_2 h_1$.*

the right, so $\mathcal{J}_{H,\psi}$ commutes with the left $\mathcal{H}$-action by convolution. Thus the left $\mathcal{H}$-action on $\mathcal{J}_{H,\psi}(C_c^\infty(I\backslash G))$ is given by the left convolution action. The result follows from the identifications

$$\mathcal{J}_{H,\psi}(C_c^\infty(I\backslash G)) \simeq \{f \in C_c^\infty(I\backslash G/H) \mid f(jgh) = f(g)\psi(h) \text{ for all } g \in G, j \in I, h \in H\}$$
$$(\operatorname{ind}_H^G \psi)^I \simeq \{f \in C_c^\infty(H\backslash G/I) \mid f(hgj) = \psi(h)f(g) \text{ for all } g \in G, j \in I, h \in H\}.$$

□

Corollary 4.11. *Assume* $(\operatorname{ind}_H^G \psi)^I \simeq R$ *as a right R-module under convolution. The functor $\mathcal{J}_{H,\psi}$ establishes an R-valued functional $\mathcal{L} : i_B^G R \to R$ which satisfies $\mathcal{L}(h\phi) = \psi(h)\mathcal{L}(\phi)$.*

Proof. Since $\mathcal{J}_{H,\psi}$ corresponds to a quotient on the right, it must commute with the left convolution action of $\mathcal{H}$ on $i_B^G R \simeq C_c^\infty(I\backslash G)$. By the above Theorem and the assumption that $(\operatorname{ind}_H^G \psi)^I \simeq R$ as a right R-module under convolution, $\mathcal{J}_{H,\psi}$ is left R-linear on $i_B^G R$. Moreover, if $h \in H$, $f \in i_B^G R$, and $g \in G$, then $\mathcal{J}_{H,\psi}(hf)(g) = \mathcal{J}_{H,\psi}(f)(gh) = \mathcal{J}_{H,\psi}(f)(g)\psi(h)$, as desired. □

At this point, we have established that model spaces of the form $\operatorname{ind}_H^G \psi$ can be used to define R-valued functionals on the universal principal series. We should also note that one can generate a space of functions from an R-valued functional $\mathcal{L}$ by $\mathcal{L}(g) := \mathcal{L}(\phi_1(g))$, and these functions will inherit any equivariance properties $\mathcal{L}$ has. However, the model space generated in this way is naturally a collection of R-valued functions. At this point, it is not clear how to associate this "R-valued model space" to a complex-valued one.

4.5 Uniqueness Criterion

In this section, we further establish our link between models of the form $\operatorname{ind}_H^G \psi$ and R-valued functionals by showing a uniqueness criterion for R-valued functions which can be computed from the model space. We begin with a uniqueness criterion for Hecke algebras which is adapted from work of Chan and Savin.

Proposition 4.12. *(Chan and Savin, [13]) For an algebra A, let $Z(A)$ denote the center of A. For any linear character ε of $\mathcal{H}_0$, there is a surjection of algebras*

$$Z(\mathcal{H}) \to \mathrm{Aut}_{\mathcal{H}}(\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon).$$

Proof. Let $\iota : \mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon \to \mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$ be an $\mathcal{H}$-module isomorphism, and let $v_\varepsilon \in \mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$ denote the "standard" copy of ε in $\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$, that is, v_ε corresponds to $1 \otimes 1$ under the isomorphism $\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon \simeq \mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon$. Then on the one hand, $T_s(\iota(\varepsilon)) = \varepsilon(T_s)\iota(v_\varepsilon)$ since ι is an automorphism of $\mathcal{H}$-modules. On the other hand, ι is an isomorphism of R-modules, so for some $r \in R$, $\iota(v_\varepsilon) = r v_\varepsilon$. Then

$$T_s(\iota(v_\varepsilon)) = r^s \varepsilon(T_s) v_\varepsilon + (1-q)\frac{r^s - r}{1 - \pi^{-\alpha^\vee}} = \varepsilon(T_s)\iota(v_\varepsilon).$$

In particular, $r^s = r$, so $r \in R^W = Z(\mathcal{H})$. Thus, the map ι is given by multiplication by an element of $Z(\mathcal{H})$. □

Combining the above work, we obtain the following Theorem.

Theorem 4.13. *Let G be a reductive group over a p-adic field. Let H be a closed subgroup of G which is exhausted by its compact open subgroups, and let ψ be a smooth character of H. Assume there is an isomorphism of $\mathcal{H}$-modules*

$$(\mathrm{ind}_H^G \psi)^I \simeq \mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$$

for a linear character ε of $\mathcal{H}_0$. Then there is an R-valued functional $\mathcal{L}_{H,\psi} : i_B^G R \to R$ satisfying $\mathcal{L}_{H,\psi}(hf) = \psi(h)\mathcal{L}_{H,\psi}(f)$ which is unique up to the center of $\mathcal{H}$.

Proof. By Corollary 4.11, such an R-valued functional $\mathcal{L}_{H,\psi}$ exists. By Theorem 4.10, the left intertwining action of $\mathcal{H}$ on the image of $\mathcal{L}_{H,\psi}$ makes the image space isomorphic to $\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$ as a left $\mathcal{H}$-module. By Proposition 4.12, this action is unique up to the center of $\mathcal{H}$. □

Chapter 5

Classification Results for R-valued functionals

In this chapter we discuss classification results for R-valued functionals on the unramified principal series. In the previous section, we saw that the $\mathcal{H}$-action on the image space of many unique R-valued functionals yields an $\mathcal{H}$-module of the form $\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$. Our first task in this chapter is to generalize the following Theorem of Chan and Savin

Theorem 5.1. *(Chan and Savin [14]) Let $\mathcal{H}$ be the Iwahori-Hecke algebra associated to $G = GL_n(\mathbb{F})$ over a nonarchimedian local field $\mathbb{F}$, and let V be any $\mathcal{H}$-module which is isomorphic to $R = \mathbb{C}[X_*]$ as an R-module. Then $V \simeq \mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon$ for a linear character ε of $\mathcal{H}$.*

We first investigate the above in the context of affine Weyl groups, where the computations are significantly easier. We then generalize the proof of Theorem 5.1. We note that Chan and Savin are primarily concerned with the convolution action of $\mathcal{H}$ on a model space, but their arguments are so algebraic in nature that these results are equally valid for the left $\mathcal{H}$-module structure on the image space of an R-valued functional.

We then build on ideas of [4] to classify all $\mathcal{H}$-module actions on the image spaces of surjective R-valued functionals (that is, functionals $\mathcal{L} : i_B^G R \to R$ such that $\mathcal{L}(\mathcal{M}) = R$). This classification relies on the functional equations

$$\mathcal{A}_s \circ \mathcal{L}(\phi) = b_s(\mathcal{L})\mathcal{L}^s(\phi)$$

for the intertwining operators $\mathcal{A}_s$. We proceed by classifying the possible "magic factors" $b_s(\mathcal{L})$ and then showing that these constants completely determine an $\mathcal{H}$-module action on R. We note that this classification does not quite classify all R-valued functionals since the magic factors $b_s(\mathcal{L})$ will not distinguish between $\mathcal{H}$-stable ideals in R.

5.1 The Affine Weyl Group Case

Here we investigate the problem of determining when an action of an extended affine Weyl group $W \ltimes X_*$ on $\mathbb{C}[X_*]$ yields a module of the form $\mathrm{ind}_W^{W \ltimes X_*} \varepsilon$. Since the Iwahori-Hecke algebra $\mathcal{H}$ is isomorphic to the group algebra of an affine Weyl group, the results we prove here are highly suggestive of the general behavior. We begin by showing that Theorem 5.1 does not without modification in this setting. Let $\widetilde{W} = S_2 \ltimes \mathbb{Z}$ be an affine Weyl group of type $\tilde{\mathcal{A}}_1$. Define a map $\mathbb{C}[\widetilde{W}] \to \mathbb{C}[\mathbb{Z}]$ by

$$\begin{aligned} v_i &\mapsto v_i \\ sv_i &\mapsto v_{1-i}. \end{aligned}$$

This is a $\widetilde{W}$-module homomorphism, as $sv_i \cdot v_j = sv_{i+j} = v_{1-i-j} = v_{-i} \cdot v_{1-j} = v_i^s s \cdot v_j$. We observe that there is no W-eigenvector in $V := \mathbb{C}[\mathbb{Z}]$ that spans $\mathbb{C}[\mathbb{Z}]$ as a $\mathbb{Z}$-module, so V is not isomorphic to an induced-from-finite $\mathbb{C}[\widetilde{W}]$-module. Moreover, V is realized by a homomorphism from the regular representation of $\widetilde{W}$, as is the situation with R-valued functionals in the $\mathcal{H}$-module case.

We can salvage the situation by recalling the Coxeter presentation of $\widetilde{W}$. Letting

s_0 be the other simple reflection in the Coxeter presentation for $\widetilde{W}$, we have $s_0v_1 = s_0(s_0s_1) = s_1 \mapsto v_1$. In particular, v_1 is fixed by s_0, and by Frobenius reciprocity we have

$$V \simeq \mathrm{ind}_{\langle s_0\rangle}^{\widetilde{W}} \mathbb{1}.$$

This adds a slight wrinkle to our program - it is possible for the module V to be isomorphic to $\mathbb{C}[X_*]$ and to be of the form $\mathrm{ind}_{W'}^{\widetilde{W}} \varepsilon$ where $W' \simeq W$ is generated by an alternate choice of simple reflections in the Coxeter presentation of $\widetilde{W}$. This observation eventually leads to our generalizations of Theorem 5.1.

We turn our attention to the rank n case. Let V be a $\mathbb{C}[\widetilde{W}]$-module, and assume there exists an R-module isomorphism $\Psi : R \to V$. As an initial observation, if $f \in R$ is a unit and $s = s_\alpha$ is a simple reflection, we have $s\Psi(f) = \Psi(gf) = g \cdot \Psi(f)$ for some $g \in R$. We also have

$$f\Psi(1) = s^2\Psi(f) = sg\Psi(f) = g^s gf\Psi(1),$$

so $g^s = g^{-1}$. Since R is isomorphic to a complex polynomial algebra, we must have $g = \lambda\pi^{\ell\alpha^\vee}$ for some $\ell \in \mathbb{Q}$ and $\lambda = \pm 1$. We use this observation together with the example above to prove the rank 1 case.

Proposition 5.2. *Let $\widetilde{W}$ be an extended affine Weyl group of type $\tilde{A}_1$, and let V be a $\widetilde{W}$-module such that $V \simeq R$ as an R-module. Then there is a simple reflection $s \in \{s_0, s_1\}$ such that*

$$V \simeq \mathrm{ind}_{\langle s\rangle}^{\widetilde{W}} \varepsilon$$

for a linear character ε of $\langle s\rangle$.

Proof. In order to reconcile notations, we identify $\pi^{\alpha^\vee}$ with (s_0s_1). By Frobenius reciprocity, it is enough to show that for some $s \in \{s_0, s_1\}$, there exists an s-eigenvector $v \in V$ which spans V as an R-module. By the above observations, we know that for any monomial $g \in R$, we have $s_1\Psi(g) = \lambda\Psi((s_0s_1)^\ell g)$ for some $\lambda \in \{\pm 1\}$ and some $\ell \in \mathbb{Z}$.

Case 1: ℓ is even, i.e. there exists some monomial $g \in R$ such that $s_1\Psi(g) = \lambda\Psi((s_0s_1)^{2n}g)$ for some $\lambda \in \{\pm 1\}$ and $n \in \mathbb{Z}$.

In this case, let $v = \Psi(\pi^{n\alpha^\vee} g) = \Psi((s_0s_1)^n g)$. Then

$$s_1 v = (s_1s_0)^n s_1 \Psi(g) = \lambda\Psi(\pi^{-n\alpha^\vee}\pi^{2n\alpha^\vee} g) = \lambda v,$$

so v is an eigenvector for s_1. $\widetilde{W}v$ must span V as $v \in \Psi(X_*)$.

Case 2: ℓ is never even, i.e. for all monomials $g \in \mathbb{R}$, $s_1\Psi(g) = \lambda\Psi((s_0s_1)^{2n-1}g)$ for some $\lambda \in \{\pm 1\}$ and $n \in \mathbb{Z}$.

Let g be any monomial in R, and let ℓ, λ, and n be as above. Let $v = \Psi(\pi^{n\alpha^\vee} g) = \Psi((s_0s_1)^n g)$. Then

$$s_0 v = (s_1s_0)^{n-1} s_1 \Psi(g) = \lambda(s_0s_1)^n \Phi(g) = \lambda v,$$

so v is an eigenvector for s_0. As above, $\widetilde{W}v$ must span V as $v \in \Psi(P^\vee)$. The result follows. □

The above Proposition holds for the usual rank 1 affine Weyl group $\widetilde{W} = \langle s_0, s_1\rangle$. However, in our applications we must also consider the extended affine Weyl group $\widetilde{W}' = \left\langle s_0, s_1, \pi^{\alpha^\vee/2} = (s_0s_1)^{1/2}\right\rangle$, where $\pi^{\alpha^\vee/2} = (s_0s_1)^{1/2}$ corresponds to the fundamental weight $\alpha^\vee/2$. We then have the following

Proposition 5.3. *Let* $\widetilde{W} = \left\langle s_0, s_1, \pi^{\alpha^\vee/2} \mid (\pi^{\alpha^\vee/2})^2 = s_0s_1, s_0^2 = s_1^2 = 1\right\rangle$ *be an extended affine Weyl group, and let* V *be a* $\widetilde{W}$*-module such that* $V \simeq R = \mathbb{C}[\pi^{\pm\alpha^\vee/2}]$ *as an* R*-module. Then* $\left\langle s_1\pi^{\alpha^\vee/2}\right\rangle \simeq \langle s_0\rangle \simeq \langle s_1\rangle$*, and there is some* $s \in \{s_1, s_1\pi^{\alpha^\vee/2}\}$ *and character* ε *of* $\langle s\rangle$ *such that* $V \simeq \operatorname{ind}_{\langle s\rangle}^{\widetilde{W}} \varepsilon$*.*

Proof. Let $g \in R$ be a monomial. Then by the above reasoning, $s_1 g = \lambda\pi^{\ell\alpha^\vee} g$. By Proposition 5.2, if $\ell \in 2\mathbb{Z}$, $V \simeq \operatorname{ind}_{\langle s_1\rangle}^{\widetilde{W}} \varepsilon$ for some character ε of s_1. We have to remaining cases.

Case 1: $\ell = 2n + 1$ for some $n \in \mathbb{Z}$.

Let $v = \Phi(\pi^{(n+\frac{1}{2})\alpha^\vee} g)$. Then $s_1 v = \pi^{(-n-\frac{1}{2})\alpha^\vee} s_1 \Phi(g) = \lambda\Phi(\pi^{(n+\frac{1}{2})\alpha^\vee} g) = \lambda v$. By Frobenius reciprocity, $V \simeq \mathrm{ind}_{\langle s_1 \rangle}^{\widetilde{W}} \varepsilon_\lambda$.

Case 2: $\ell = n + \frac{1}{2}$ for some $n \in \mathbb{Z}$.

Let $v = \Phi(\pi^{\frac{n}{2}\alpha^\vee} g)$. Then $s_1 \pi^{\alpha^\vee/2} v = \pi^{(-\frac{n}{2}-\frac{1}{2})\alpha^\vee} s_1 \Phi(g) = \pi^{(-\frac{n}{2}-\frac{1}{2})\alpha^\vee} \lambda \Phi(\pi^{(n+\frac{1}{2})\alpha^\vee} g) = \lambda v$. It follows by Frobenius reciprocity that $V \simeq \mathrm{ind}_{\langle s_1 \pi^{\alpha^\vee/2} \rangle}^{\widetilde{W}} \varepsilon_\lambda$. $\square$

In these cases we were able to identify copies W' of the finite Weyl group such that a given vector v_i spans a one dimensional representation of W', thus identifying our space with a $\widetilde{W}$-representation of the form $\mathrm{ind}_{W'}^{\widetilde{W}} \varepsilon$. Before we generalize this property, we show the abundance of such subgroups W'.

Lemma 5.4. *Let $\widetilde{W} = W \ltimes X_*$, and let $\ell_1, \ldots, \ell_n$ be rational numbers such that $\ell_i \alpha_i^\vee \in X_*$. The elements $s_i' = \pi^{\ell_i \alpha_i^\vee} s_i$ all have order two, and there is an isomorphism*

$$W \simeq \langle s_1', \ldots, s_n' \rangle .$$

Proof. First, we note that elements of the form $s_i' = \pi^{\ell_i \alpha_i^\vee} s_i$ have order two:

$$(\pi^{\ell_i \alpha_i^\vee} s_i)^2 = \pi^{\ell_i \alpha_i^\vee} \pi^{-\ell_i \alpha_i^\vee} = 1.$$

We also note that if $s_{i_1}' \cdots s_{i_r}' = 1$ is a relation in $\left\langle s_1', \ldots, s_n' \right\rangle$, then conjugating all of the translation elements to one side of the product yields an element of the form

$$s_{i_1} \cdots s_{i_r} \pi^{\beta^\vee}$$

for some $\beta^\vee \in X_*$. In particular, $s_{i_1} \cdots s_{i_r} = 1$ is a relation in W. Now assume $(s_i s_j)^m =$

1. Then we have

$$\begin{aligned}(s_i's_j')^m &= \underbrace{(\pi^{\ell_i\alpha_i^\vee}s_i\pi^{\ell_j\alpha_j^\vee}s_j)\cdots(\pi^{\ell_i\alpha_i^\vee}s_i\pi^{\ell_j\alpha_j^\vee}s_j)}_{m\text{ factors}}\\ &= s_i\pi^{\ell_i(1+(s_is_j)+\cdots+(s_is_j)^{m-1})\alpha_i^\vee}(s_is_j)^{m-1}\pi^{\ell_j(1+(s_js_i)+\cdots+(s_js_i)^{m-1})\alpha_j^\vee}s_j.\end{aligned}$$

Consider the term $(1+(s_is_j)+\cdots+(s_is_j)^{m-1})\alpha_i^\vee$ in the first exponent as an element in the reflection representation of $\langle s_i, s_j\rangle$. This element must be fixed by the Coxeter element s_is_j. Since the Coxeter element only fixes 0 in the reflection representation, we must have $(1+(s_is_j)+\cdots+(s_is_j)^{m-1})\alpha_i^\vee = 0$. Similarly, we get $\ell_j(1+(s_js_i)+\cdots+(s_js_i)^{m-1})\alpha_j^\vee = 0$. Thus, the above expression reduces to

$$(s_i's_j')^m = 1,$$

as desired. It follows that the group $\langle s_1', \ldots, s_n'\rangle$ satisfies all of the braid relations satisfied in W and none others. The result follows. $\square$

Proposition 5.5. *Let V be a $\widetilde{W}$-module which is isomorphic to R as an R-module. Then there is an isomorphic copy W' of W in $\widetilde{W}$ such that $\widetilde{W} = X_* \rtimes W'$ and $V \simeq \operatorname{ind}_{W'}^{\widetilde{W}} \varepsilon$ for a linear character ε of W'.*

Proof. Let g be a monomial in R. Let $s_i \in S$, and consider the rank 1 affine Weyl group generated by s_i and $c\alpha_i^\vee$, where $c \in Q$ is the smallest number such that $c\alpha_i^\vee \in X_*$. Then $s_i\Phi(g) = \lambda_i\pi^{\ell_i\alpha_i^\vee}\Psi(g)$ for some $\ell_i \in \mathbb{Q}$ and $\lambda_i \in \mathbb{C}$. For each i, define $s_i' = \pi^{-\ell_i\alpha_i^\vee}s_i$. By Lemma 5.4, $W \simeq W' = \langle s_1', \ldots, s_n'\rangle$, and by construction, $s_i'\Psi(g) = \lambda_i\Psi(g)$. Then $\Psi(g)$ spans a one dimensional representation of W'. As $\widetilde{W}\Psi(g)$ spans V, the result follows by Frobenius reciprocity. $\square$

We observe that, in many cases, the subgroup W' is conjugate to W, and otherwise, there may be a small number of distinct conjugacy classes of subgroups $W' \simeq W$. This is relevant, since if $\Psi(g)$ is a simultaneous eigenvector for each $s_i' \in W_i$ and $W^x = W'$, then

$$s_ix\Psi(g) = xs_i'\Psi(g) = \lambda_ix\Psi(g).$$

In particular, $x\Psi(g)$ is a simultaneous eigenvector for each simple reflection $s_i \in W$. We have the following:

Theorem 5.6. *Let V be a $\widetilde{W}$-module which is isomorphic to R as an R-module. Then V is of the form* $\mathrm{ind}_{W'}^{\widetilde{W}} \varepsilon$ *for some subgroup $W' \simeq W$ and some linear character ε of W'.*

A natural question to ask at this point is how many subgroups W' must be considered. As inner automorphisms always give isomorphic representations, we need only consider subgroups W' which are not conjugate to each other. Since for any simple reflection s_i, $\pi^{-\alpha_i^\vee} s_i \pi^{\alpha_i^\vee} = s_i \pi^{2\alpha_i^\vee}$, we can always shift the constants ℓ_i by 2 (or 1, if $\alpha_i/2 \in X_*$). It follows that there is a finite collection of subgroups subgroups $W_1, \ldots, W_m$ of $\widetilde{W}$ which must be considered in Theorem 5.6.

Relatedly, one may wonder when we don't need to consider copies of W at all. This is the case for $G = GL_n$, as shown in the Hecke algebra case in [14]. The proof of this property involves solving a system of equations in $\mathbb{C}[X_*]$. In particular, the group G must have a connected center, but this is not sufficient to conclude that all $\mathbb{C}[\widetilde{W}]$ modules which are isomorphic to R are of the form $\mathrm{ind}_W^{\widetilde{W}} \varepsilon$.

5.2 The Hecke Algebra Case

In this section, we study the Hecke algebra version of the above result, following the techniques from Chan and Savin. As in the affine Weyl group case, we begin by studying the rank 1 example, where $W = S_2 = \langle s_1 \rangle$. We wish to study the actions of T_0 and T_1 on $\mathrm{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$ for some character ε of $\mathcal{H}_0$. Using the definition of $\mathcal{H}$ as a Hecke algebra associated to the Coxeter group $\tilde{A}_1$, we see that the map $T_0 \leftrightarrow T_1$ is an isomorphism of $\mathcal{H}$. In particular, the desired computation is equivalent to computing the actions of T_0 and T_1 on $\mathrm{ind}_{\langle T_0 \rangle}^{\mathcal{H}} \varepsilon$. While we have a version of the Bernstein relation for T_0, it is not immediately compatible with the isomorphism $T_0 \leftrightarrow T_1$.

The action of T_1 on $\operatorname{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$ is given by the Bernstein relation:

$$T_1(\pi^{n\alpha^\vee}) = \pi^{-n\alpha^\vee}\varepsilon(T_1) + (1-q)\frac{\pi^{-n\alpha^\vee} - \pi^{n\alpha^\vee}}{1-\pi^{-\alpha^\vee}}.$$

Using the Coxeter presentation, we have

$$\begin{aligned}
T_0\pi^{n\alpha^\vee} = T_0(\tfrac{1}{q}T_0T_1)^n &= \tfrac{1}{q}(q+(q-1)T_0)T_1\pi^{(n-1)\alpha^\vee} \\
&= T_1\pi^{(n-1)\alpha^\vee} + (q-1)\pi^{n\alpha^\vee} \\
&= \pi^{-(n-1)\alpha^\vee}\varepsilon(T_1) + (1-q)\frac{\pi^{(-n+1)\alpha^\vee} - \pi^{(n-1)\alpha^\vee}}{1-\pi^{-\alpha^\vee}} + (q-1)\pi^{n\alpha^\vee} \\
&= \pi^{-(n-1)\alpha^\vee}\varepsilon(T_1) + (1-q)\frac{\pi^{(-n+1)\alpha^\vee} - \pi^{n\alpha^\vee}}{1-\pi^{-\alpha^\vee}}
\end{aligned}$$

Under the map $T_0 \leftrightarrow T_1$, we replace $\pi^{\alpha^\vee}$ with $\frac{1}{q}T_1T_0$. We express T_1T_0 in terms of the standard generators of $\mathcal{H}$ like so:

$$\begin{aligned}
T_1T_0 = T_1T_0(T_1T_1^{-1}) &= T_1\pi^{\alpha^\vee}T_1^{-1} \\
&= (\pi^{-\alpha^\vee}T_1 + (q-1)(-1-\pi^{\alpha^\vee}))T_1^{-1} \\
&= \pi^{-\alpha^\vee} + \tfrac{1-q}{q}(1+\pi^{\alpha^\vee})(T_1 + (1-q)).
\end{aligned}$$

The identity

$$T_1\pi^{-\alpha^\vee} = T_1(qT_1^{-1}T_0^{-1}) = qT_0^{-1} = T_0 + 1 - q.$$

is quite helpful in computing the structure of $\operatorname{ind}_{\langle T_0\rangle}^{\mathcal{H}} \varepsilon$. Using the above together with the Bernstein relations, we get

$$\begin{aligned}
T_1(\pi^{n\alpha^\vee}) = T_1\pi^{(n+1)\alpha^\vee}\pi^{-\alpha^\vee} &= \pi^{(-n-1)\alpha^\vee}T_1(\pi^{-\alpha^\vee}) + (1-q)\frac{\pi^{(-n-1)\alpha^\vee} - \pi^{(n+1)\alpha^\vee}}{1-\pi^{-\alpha^\vee}}\pi^{-\alpha^\vee} \\
&= \pi^{(-n-1)\alpha^\vee}(\varepsilon(T_0) + 1 - q) + (1-q)\frac{\pi^{(-n-2)\alpha^\vee} - \pi^{n\alpha^\vee}}{1-\pi^{-\alpha^\vee}} \\
&= \pi^{(-n-1)\alpha^\vee}\varepsilon(T_0) + (1-q)\frac{\pi^{(-n-1)\alpha^\vee} - \pi^{n\alpha^\vee}}{1-\pi^{-\alpha^\vee}}.
\end{aligned}$$

In particular, none of the monomials $\pi^{n\alpha^\vee}$ in $R \simeq \mathrm{ind}_{\langle T_0\rangle}^{\mathcal{H}} \varepsilon$ are eigenvectors for the $\langle T_1\rangle$ action. As in the affine Weyl group case, inducing linear characters of $\langle T_0\rangle$ and $\langle T_1\rangle$ yields non-isomorphic $\mathcal{H}$-modules.

In light of this example, we define a family of outer automorphisms of $\mathcal{H}$. Let $P^\vee$ denote the coweight lattice of G. The injection $X_* \hookrightarrow P^\vee$ induces an injection of Hecke algebras from $\mathcal{H}$ to an extended affine Hecke algebra associated to the extended affine Weyl group $W \ltimes P^\vee$. For any coweight $\beta \in P^\vee$, there is an automorphism $\tau_\beta(T) = \pi^\beta T \pi^{-\beta}$ of the larger Hecke algebra. Such an automorphism clearly fixes $\mathbb{C}[X_*]$, and for any simple reflection s_α, we have

$$\tau_\beta(T_{s_\alpha}) = \pi^{\langle\beta,\alpha\rangle\alpha^\vee}T_{s_\alpha} + (1-q)\frac{\pi^{\langle\beta,\alpha\rangle\alpha^\vee} - 1}{1-\pi^{-\alpha^\vee}} \in \mathcal{H}.$$

Thus, τ_β gives an automorphism of $\mathcal{H}$. Combining our understanding of the outer automorphisms τ_β with the reasoning in [14] yields a general criterion for the $\mathcal{H}$-action on the image of an R-valued functional to be isomorphic to an $\mathcal{H}$-module induced from a linear character on a copy of $\mathcal{H}_0$ in $\mathcal{H}$.

Theorem 5.7. *Assume that for each simple root α, the cocharacter lattice X_* does not contain $\alpha^\vee/2$. Then every $\mathcal{H}$-module which is isomorphic to R as an R-module is of the form $\mathrm{ind}_{\tau_\beta(\mathcal{H}_0)}^{\mathcal{H}} \varepsilon$ for a linear character ε of $\mathcal{H}_0$ and a coweight β.*

Proof. Let V be an $\mathcal{H}$-module which is isomorphic to R as an R-module. By Frobenius reciprocity, it is sufficient to identify a simultaneous eigenvector $f_0 \in V$ for an appropriate copy $\tau(\mathcal{H}_0)$ of $\mathcal{H}_0$ inside $\mathcal{H}$ such that f_0 generates V as an R-module. Let s_α be a simple reflection, and let $f \in R^\times$. Then there is some $g \in R$ such that $T_{s_\alpha}(f) = gf$. On one hand, we have

$$T_{s_\alpha}^2(f) = T_{s_\alpha}(gf) = \left[g^{s_\alpha} g + (1-q)\frac{g^{s_\alpha} - g}{1 - \pi^{-\alpha^\vee}}\right] f.$$

Applying the quadratic relation to $T_{s_\alpha}^2(f)$ and re-arranging yields the identity

$$g^{s_\alpha} g = (q-1)\left[g + \frac{g^{s_\alpha} - g}{1 - \pi^{-\alpha^\vee}}\right] + q = (q-1)\frac{g^{s_\alpha} - \pi^{-\alpha^\vee} g}{1 - \pi^{-\alpha^\vee}} + q. \tag{5.1}$$

By considering the degree of both expressions, g must be a polynomial in $\pi^{\alpha^\vee}$ or $\pi^{-\alpha^\vee}$. Write $g(\pi^{\alpha^\vee}) = c_n \pi^{n\alpha^\vee} + \cdots + c_0$. Then the left hand side of (5.1) becomes

$$g^{s_\alpha} g = c_n c_0 \pi^{n\alpha^\vee} + (c_n c_1 + c_{n-1} c_0)\pi^{(n-1)\alpha^\vee} + \cdots + (c_n^2 + c_{n-1}^2 + \dots) + \cdots + c_0 c_n \pi^{-n\alpha^\vee} + q,$$

and if $n > 0$, the right hand side is

$$(1-q)(c_n \pi^{(n-1)\alpha^\vee} + (c_n + c_{n-1})\pi^{(n-2)\alpha^\vee} + \cdots + (c_0 + \cdots + c_n) + \cdots + c_n \pi^{(1-n)\alpha^\vee}) + q.$$

Equating coefficients yields $c_0 = 0$ and $c_{n-1} = c_{n-2} = \cdots = c_1 = 1 - q$, and c_n is a root of $c_n^2 + (q-1)c_n - q$, so $c_n = \lambda \in \{1, -q\}$. Then

$$g = (1-q)(\pi^{\alpha^\vee} + \cdots + \pi^{n\alpha^\vee}) - \lambda \pi^{n\alpha^\vee}.$$

Similarly, when $n < 0$ we find

$$g = (q-1)(1 + \pi^{-\alpha^\vee} + \cdots + \pi^{(n+1)\alpha^\vee}) + \lambda \pi^{n\alpha^\vee}.$$

It follows that for each simple root α there is an integer n_α and a scalar $\lambda_\alpha \in \{-1, q\}$

such that

$$T_{s_\alpha}(f) = g_\alpha f = \left[\lambda_\alpha \pi^{n_\alpha \alpha^\vee} + (1-q)\frac{\pi^{n_\alpha \alpha^\vee} - 1}{1 - \pi^{-\alpha^\vee}}\right] f. \tag{5.2}$$

By assumption, there is a coweight β such that for each simple root α, $\langle \beta, \alpha \rangle = n_\alpha$. Then for each simple root,

$$\begin{aligned}\pi^{-\beta} T_{s_\alpha} \pi^{\beta^\vee}(f) &= \left[\pi^{-\langle \beta,\alpha\rangle \alpha^\vee} T_\alpha + (1-q)\frac{\pi^{-\langle \beta,\alpha\rangle \alpha^\vee} - 1}{1-\pi^{-\alpha^\vee}}\right] f \\ &= \left[\pi^{-n_\alpha \alpha^\vee} g + (1-q)\frac{\pi^{-n_\alpha \alpha^\vee} - 1}{1-\pi^{-\alpha^\vee}}\right] f = \lambda_\alpha f.\end{aligned}$$

Thus, f is a simultaneous eigenvector for $\tau_\beta(\mathcal{H}_0)$, as desired. □

At this point, one may wonder if it is possible to prove a Hecke algebra version of Theorem 5.6. In the case of $G = PGL_2$, the cocharacter lattice contains $\alpha^\vee/2$. In the spirit of our affine Weyl group constructions, we consider the algebra generated by $Y := T_s \pi^{-\alpha^\vee/2}$. Y satisfies a quadratic relation, although not the standard one:

$$\begin{aligned}Y^2 &= T_s \pi^{-\alpha^\vee/2} T_s \pi^{-\alpha^\vee/2} \\ &= T_s \pi^{-\alpha^\vee/2}\left[\pi^{\alpha^\vee/2} T_s + (1-q)\frac{\pi^{\alpha^\vee/2} - \pi^{-\alpha^\vee/2}}{1-\pi^{-\alpha^\vee}}\right] \\ &= T_s^2 + (1-q)T_s = q.\end{aligned}$$

Thus, Y generates a two dimensional algebra, and has characters $\pm\sqrt{q}$. This element Y will help us explain some of the results in our classification of unique R-valued functionals on $\mathcal{M}$. It would be interesting to study the $|W|$-dimensional subalgebras of $\mathcal{H}$ in a more systematic way in the future, possibly leading to a Hecke algebra version of Theorem 5.6.

5.3 Classification of Surjective R-Valued Functionals

In this section we focus on the $\mathcal{H}$-module action on the image of an R-valued functional. As above, we begin with the rank 1 case. Let $\mathcal{L}$ be a surjective R-valued functional, and define

$$b_s(\mathcal{L}) := (\mathcal{L} \circ \mathcal{A}_s)/\mathcal{L}^s,$$

the constant from the functional equation associated to the intertwining operator $\mathcal{A}_s$. One direction of the following Theorem was proven by Brubaker, Bump, and Friedberg in an unpublished document.

Theorem 5.8. *Suppose that $b_s(\mathcal{L})$ is in R.*[1] *$b_s(\mathcal{L})$ is a solution to*

$$X_s^s X_s = (1 - q^{-1}\pi^{\alpha^\vee})(1 - q^{-1}\pi^{-\alpha^\vee}), \tag{5.3}$$

and all solutions to (5.3) in R uniquely define an $\mathcal{H}$-action on R in which R acts by translation and T_s acts by the "demazure-like" operator

$$T_s \ : \ f \longmapsto \frac{q}{1 - \pi^{\alpha^\vee}} \left[X_s f^s + (q^{-1} - 1)\pi^{\alpha^\vee} f \right] \tag{5.4}$$

Proof. We begin by showing that the operator (5.4) satisfies the quadratic relation if

[1] *As opposed to a localization of R*

and only if (5.3) holds. On the one hand,

$$\begin{aligned}
T_s \circ T_s(f) =& T_s\left(\frac{qX_s}{1-\pi^{\alpha^\vee}} f^s + \frac{q(q^{-1}-1)\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}} f\right)\\
=& \frac{q}{1-\pi^{\alpha^\vee}}\left[X_s\left(\frac{qX_s}{1-\pi^{\alpha^\vee}} f^s + \frac{q(q^{-1}-1)\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}} f\right)^s\right.\\
&\left. +(q^{-1}-1)\pi^{\alpha^\vee}\left(\frac{qX_s}{1-\pi^{\alpha^\vee}} f^s + \frac{q(q^{-1}-1)\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}} f\right)\right]\\
=& \frac{q}{1-\pi^{\alpha^\vee}}\left[\left(\frac{qX_sX_s^s}{1-\pi^{-\alpha^\vee}} + (q^{-1}-1)\pi^{\alpha^\vee}\left(\frac{q(q^{-1}-1)\pi^{\alpha^\vee}}{1-\pi^{\alpha^\vee}}\right)\right) f\right.\\
&\left. +\left((q^{-1}-1)\pi^{\alpha^\vee}\frac{qX_s}{1-\pi^{\alpha^\vee}} + \frac{q(q^{-1}-1)\pi^{-\alpha^\vee}}{1-\pi^{-\alpha^\vee}}\right) f^s\right]\\
=& \frac{q}{1-\pi^{\alpha^\vee}}\left[\left(\frac{q}{1-\pi^{-\alpha^\vee}}X_sX_s^s + \frac{q(q^{-1}-1)^2\pi^{2\alpha^\vee}}{1-\pi^{\alpha^\vee}}\right) f + \frac{(1-q)\pi^{\alpha^\vee}X_s - (1-q)}{1-\pi^{\alpha^\vee}} f^s\right]\\
=& \frac{q}{1-\pi^{\alpha^\vee}}\left[\left(\frac{q}{1-\pi^{-\alpha^\vee}}X_sX_s^s + \frac{q(q^{-1}-1)^2\pi^{2\alpha^\vee}}{1-\pi^{\alpha^\vee}}\right) f + (q-1)X_sf^s\right],
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
(q-1)T_s(f) + qf &= \frac{q(q-1)}{1-\pi^{\alpha^\vee}}\left[X_sf^s + (q^{-1}-1)\pi^{\alpha^\vee} f + \frac{1-\pi^{\alpha^\vee}}{(q-1)} f\right]\\
&= \frac{q}{1-\pi^{\alpha^\vee}}\left[\left((1-q-q^{-1})\pi^{\alpha^\vee}+1\right) f + (q-1)X_sf^s\right].
\end{aligned}$$

In particular, both the action of T_s^2 and of $(q-1)T_s + q$ on an arbitrary element f in R result in terms multiplying f and terms in R multiplying f^s on both sides. The coefficients of f^s for $T_s^2(f)$ and for $(q-1)T_s(f) + qf$ match for any element $X_s \in R$. Thus, these match if and only if

$$\frac{q}{1-\pi^{-\alpha^\vee}}X_sX_s^s + \frac{q(q^{-1}-1)^2\pi^{2\alpha^\vee}}{1-\pi^{\alpha^\vee}} = (1-q-q^{-1})\pi^{\alpha^\vee} + 1.$$

Simplifying,

$$\begin{aligned}
X_s X_s^s &= \frac{1-\pi^{-\alpha^\vee}}{q}\left((1-q-q^{-1})\pi^{\alpha^\vee}+1-\frac{q(q^{-1}-1)^2\pi^{\alpha^\vee}}{\pi^{-\alpha^\vee}-1}\right)\\
&= (q^{-1}-1-q^{-2})(\pi^{\alpha^\vee}-1)+q^{-1}(1-\pi^{-\alpha^\vee})+(q^{-1}-1)^2\pi^{\alpha^\vee}\\
&= -q^{-1}\pi^{\alpha^\vee}-q^{-1}\pi^{-\alpha^\vee}+1+q^{-2}\\
&= (1-q^{-1}\pi^{\alpha^\vee})(1-q^{-1}\pi^{-\alpha^\vee}),
\end{aligned}$$

so our operator satisfies the quadratic relation if and only if (5.3) holds.

We now verify that the Demazure-like action of T_s is consistent with the Bernstein relations. Let f and g be in R. Then

$$T_s(fg) = \frac{q}{1-\pi^{\alpha^\vee}}\left[X_s f^s g^s + (q^{-1}-1)\pi^{\alpha^\vee} fg\right].$$

On the other hand,

$$\begin{aligned}
T_s(fg) &= f^s T_s(g) + (1-q)\frac{f^s-f}{1-\pi^{-\alpha^\vee}}g\\
&= f^s\left(\frac{q}{1-\pi^{\alpha^\vee}}\right)\left(X_s g^s + (q^{-1}-1)\pi^{\alpha^\vee} g\right) + (1-q)\frac{f^s-f}{1-\pi^{-\alpha^\vee}}g\\
&= \frac{q}{1-\pi^{\alpha^\vee}}\left[X_s f^s g^s + (q^{-1}-1)\pi^{\alpha^\vee} fg\right],
\end{aligned}$$

as desired.

Finally, to see that the factors $b_s(\mathcal{L})$ satisfy the relation (5.3), we recall the relation between the normalized intertwining operators $\mathcal{A}_s$ and the Hecke algebra generators T_s. The Demazure-like action of T_s on R defined with $X_s = b_s(\mathcal{L})$ necessarily defines a Hecke action on the image of $\mathcal{L}$ in R, as in Lemma 4.3. In particular, this operator satisfies the quadratic relation for T_s, so $b_s(\mathcal{L})$ satisfies (5.3).

□

This Theorem has two main applications - it provides a restriction on the factors

$b_s(\mathcal{L})$, and it allows us to algebraically define an R-valued functional on $\mathcal{M}$ without explicit reference to any model space for G. Solutions to (5.3) allow us to exhaust the world of surjective R-valued functionals on $\mathcal{M}$. With this motivation in mind, We turn our attention to solutions to $X_s^s X_s = (1 - q^{-1}\pi^{\alpha^\vee})(1 - q^{-1}\pi^{-\alpha^\vee})$. We observe that if X_s is a solution to this equation, so is $\pi^{n\alpha^\vee} X_s$. Note that automorphisms τ in the style of Theorem 5.7 always produce such monomial shifts. Moreover, those shifts realized by inner automorphism are the squares of the cocharacters, which form a finite index set in X_*. Thus only finitely many monomial shifts need to be considered. As these shifts do not alter the following computations, we suppress them here. Disregarding all monomial shifts,[2] there are six solutions to the rigidity equation, summarized in the table below:

$X = b_s(\mathcal{L})$	$\mathcal{H}$-module
$\pi^{\alpha^\vee} - q^{-1}$	$\mathrm{Ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon_{sgn}$
$1 - q^{-1}\pi^{\alpha^\vee}$	$\mathrm{Ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon_{triv}$
$(1 - q^{-1/2}\pi^{\alpha^\vee/2})(1 + q^{-1/2}\pi^{-\alpha^\vee/2})$	$\mathrm{Ind}_{\langle T_s\pi^{-\alpha^\vee/2}\rangle}^{\mathcal{H}} \varepsilon_{\sqrt{q}}$
$(1 - q^{-1/2}\pi^{-\alpha^\vee/2})(1 + q^{-1/2}\pi^{\alpha^\vee/2})$	$\mathrm{Ind}_{\langle T_s\pi^{-\alpha^\vee/2}\rangle}^{\mathcal{H}} \varepsilon_{-\sqrt{q}}$
$-(1 - q^{-1/2}\pi^{\alpha^\vee/2})(1 + q^{-1/2}\pi^{-\alpha^\vee/2})$	$(\mathrm{Ind}_{\langle T_s\pi^{-\alpha^\vee/2}\rangle}^{\mathcal{H}} \varepsilon_{\sqrt{q}})(\pi^{\alpha^\vee/2} - \pi^{-\alpha^\vee/2})$
$-(1 - q^{-1/2}\pi^{-\alpha^\vee/2})(1 + q^{-1/2}\pi^{\alpha^\vee/2})$	$(\mathrm{Ind}_{\langle T_s\pi^{-\alpha^\vee/2}\rangle}^{\mathcal{H}} \varepsilon_{-\sqrt{q}})(\pi^{\alpha^\vee/2} - \pi^{-\alpha^\vee/2})$

It is an interesting open problem to attribute linear functionals to each of the rows of the above table. It is known that the first line in the above table corresponds to the Whittaker or split special-orthogonal Bessel models [4] and the second corresponds to the spherical [3] and non-split Waldspurger models [6]. The final four $\mathcal{H}$-modules merit some further discussion. The first key observation is that there are "extra" elements Y of $\mathcal{H}$ which generate two-dimensional subalgebras and for which $\mathcal{H} = \langle Y \rangle \ltimes R$. As we previously observed, taking $Y = T_s\pi^{-\alpha^\vee/2}$, we have $Y^2 = (T_s\pi^{-\alpha^\vee/2})^2 = q$. Then $\langle Y \rangle$ has two linear characters corresponding to $Y \mapsto \sqrt{q}$ and $Y \mapsto -\sqrt{q}$. The remaining two

[2] Exactly which shifts correspond to inner automorphisms will depend on the root datum of the group.

modules can be explained by identifying R with the principal ideal $(\pi^{\alpha^\vee/2} - \pi^{-\alpha^\vee/2})R$. Indeed, if V is an $\mathcal{H}$-module of the form described in Lemma 4.3, $(\pi^{\alpha^\vee/2} - \pi^{-\alpha^\vee/2})V$ is also such an $\mathcal{H}$-module with $b_s((\pi^{\alpha^\vee/2} - \pi^{-\alpha^\vee/2})V) = -b_s(V)$. The third line of the above table can be realized on SL_2 using a period integral built from the Whittaker-Shintani functional as studied in [42]. Finally, Sakellaridis [38] has studied matrix coefficients from the perspective of spherical varieties, and makes a list of factors $b_s(\mathcal{L})$ arising from the finite list of cases for rank one spherical varieties. The reader should compare the results in Section 5.2 of [38] to the table above.

Since any Iwahori-Hecke algebra is generated by finitely many rank 1 Iwahori-Hecke algebras the above shows that there are finitely many surjective, unique R-valued functionals in general. Thus, we have the following.

Theorem 5.9. *Let G be a reductive group over a non-Archimedean local field, and let $\mathcal{M}$ be as above. There are finitely many unique, surjective R-valued functionals $\mathcal{L} : \mathcal{M} \to R$. Moreover, if $\mathcal{L}'$ is any unique R-valued functional on $\mathcal{M}$, there is a unique, surjective R-valued functional $\mathcal{L}$ on $\mathcal{M}$ and an embedding of left $\mathcal{H}$-modules*

$$\mathcal{L}'(\mathcal{M}) \hookrightarrow \mathcal{L}(\mathcal{M}).$$

In particular, the possible $\mathcal{H}$-modules obtained by the left action on the image of a unique R-valued functional are realized as $\mathcal{H}$-stable ideals in an $\mathcal{H}$-module determined by a solution to (5.3).

We point out that adjacent simple roots of the same length must have the same coefficient X_s. Thus, there are in general many fewer than $2^{\operatorname{rank} G}$ possible surjective functionals for a given group.

Chapter 6

Generalized Gelfand-Graev Representations over p-adic Fields

In this section we discuss how the machinery developed so far can be adapted to the study of reductive groups over p-adic fields in general and the GL_n case in particular. The construction of the generalized Gelfand-Graev representation is essentially unchanged over a p-adic field $\mathbb{F}$. In this situation, we focus on finding good models for representations with Iwahori fixed vectors which may not be captured by the above theory. In analogy to the finite field setting above, we study such representations by studying the space of I-fixed vectors as modules of the Iwahori-Hecke algebra $\mathcal{H} := C_c^\infty(G//I)$. We conclude the section by discussing generalizations of the theory developed in the previous chapter to generalized Gelfand-Graev representations.

6.1 The gGGr Construction over p-Adic Fields

This section reviews the generalized Gelfand-Graev construction for reductive groups over a p-adic field $\mathbb{F}$, where p is a large prime. In this context, it is common to refer to the gGGr construction as a "generalized Whittaker model" in the style of Moeglin and Waldspurger [33], although for internal consistency we will avoid this term. As mentioned in our discussion of gGGrs over finite fields, the Dynkin-Kostant classification of nilpotent orbits in terms of weighted Dynkin diagrams holds over $\mathbb{F}$, as does the Kirillov orbit method. Thus, given a representative X of some nilpotent orbit in $\mathfrak{g}$, we can pick a neutral element H in a standard triple containing X as a nilpositive and grade $\mathfrak{g}$ by $\operatorname{ad} H$ eigenvalues. As before, we define the subgroup U_1 generated by exponentiating the positive $\operatorname{ad} H$ eigenspaces. Fixing a character χ of $\mathbb{F}$ trivial on $\mathfrak{p}$ but not on $\mathfrak{o}$, we can apply the Kirillov construction to $\chi \circ X_\vee$ to produce an irreducible U_1 representation η. We then have the gGGr

$$\Gamma_X = \operatorname{ind}_{U_1}^G \eta,$$

where ind denotes compactly supported smooth induction.[1] Also as before, we can let $U_X = U_{1.5}$ be the exponentiation of a maximal subordinate subalgebra $\mathfrak{u}_{1.5} \subset \mathfrak{u}_1$ with respect to $X^\vee$. By a transitivity of induction argument from [24], we can define $\Gamma_X = \operatorname{ind}_{U_{1.5}}^G \psi_X$ where $\phi_X(\exp u) := \exp(\chi \circ X^\vee(u))$ for all $u \in \mathfrak{u}_{1.5}$. This description is much easier to work with in this case, as η_X is infinite dimensional and ψ_X is one dimensional. We assume the character ψ_X has conductor $\mathfrak{p}$.

While it is possible to define the refined generalized Gelfand-Graev representation for $G(\mathbb{F})$, the story here is more delicate than in the finite case. Given any subgroup S of $Z_L([\eta])$ and representation ρ of S, we can still associate elements of S to intertwining operators on η. Unlike in the finite case, in the p-adic setting η will in general be projective as an S-module. When discussing refined gGGrs in this case, Kawanaka chooses projective representations ρ of $Z_L([\eta])$ so that $\rho \otimes \tilde{\eta}$ is a genuine representation of $Z_L([\eta])U_1$, and then the refined gGGrs are defined as $\Gamma_{O_X,\rho} \operatorname{ind}_{Z_L([\eta])U_1}^G \rho \otimes \tilde{\eta}$. By [47], we have $Z_L([\eta]) = Z_L(X)$. When $Z_L(X)$ normalizes $U_{1.5}$ we can construct the

[1] By a comment in [24], one should be able to rephrase most results about Γ_X using unrestricted induction. However, this simplifying assumption is common and makes the relevant analysis significantly easier, so we will follow other authors in making it here.

representations

$$\tilde{\Gamma}_{O_X,\rho} := \mathrm{ind}_{Z_L(X)U_{1.5}}^G \rho \otimes \phi.$$

While Kawanaka does not make this specification, we have found that replacing $Z_L(X)$ with its maximal compact subgroup makes the p-adic analysis easier in a variety of ways.

6.2 Relation to the Broader Theory

In a program originating with Moeglin and Waldspurger in 1987, generalized Gelfand-Graev representations for $G(\mathbb{F})$ can be studied via the "wave front" of smooth, admissible, irreducible representations of G. In this section we provide a brief description of the objects involved in this program in order to present a recent result by Gomez, Gourevich, and Sahi about the structure of gGGrs for $GL_n(\mathbb{F})$. The wave front made its debut in a 1973 paper by Harish-Chandra [26], and we begin by following his construction. Let π be an irreducible, smooth, admissible representation of $G(\mathbb{F})$. The *distribution character* on π is the map $\Theta_\pi : C_c^\infty(G) \to \mathbb{C}$ given by

$$\Theta_\pi(f) := \mathrm{Tr} \int_G f(x)\pi(x)dx.$$

It can be shown that $\int_G f(x)\pi(x)dx$ is always of finite rank, so Θ_π is indeed well defined. Fixing a nontrivial character $\chi : \mathbb{F} \to \mathbb{C}$, we can define a "Fourier transform" on $C_c^\infty(\mathfrak{g})$ by

$$\mathcal{F}(f)(Y) := \int_{\mathfrak{g}} f(X)\chi(\kappa(X,Y))dX,$$

where κ denotes the Killing form on $\mathfrak{g}$. By [37], for each nilpotent orbit O, we can define a G-invariant measure on O by

$$\mu_O(f) = \int_{G/G_X} f(\mathrm{Ad}(y)X)dy^*,$$

where dy^* is the measure on the quotient space G/G_X derived from the Haar measure on G. The Fourier transform on these measures is given by $\mathcal{F}(\mu_O)(f) := \mu_O(\mathcal{F}(f))$. In 1973, Harish-Chandra showed that Fourier transforms of these measures has a surprising connection to Θ_π:

Theorem 6.1. *(Harish-Chandra, [26]) Let π be an irreducible smooth representation of G, and let $\mathcal{N}$ denote the collection of nilpotent orbits in $\mathfrak{g}$. Then asymptotically near the identity,*

$$\exp^* \Theta_\pi \sim \sum_{O \in \mathcal{N}} c_O(\pi) \mathcal{F}(\mu_O)$$

for some complex numbers $c_O(\pi)$.

Motivated by the above, we define the *wave front* of π as

$$WF(\pi) = \overline{\bigcup_{\substack{O \in \mathcal{N} \\ c_O(\pi) \neq 0}} O},$$

where the closure in definition of $WF(\pi)$ is in the $\mathbb{F}$ topology. An early result explaining the connection of $WF(\pi)$ to gGGrs is as follows.

Theorem 6.2. *(Moeglin and Waldspurger [33]) Let π be a smooth, admissible, irreducible representation of $G(\mathbb{F})$, and let $\mathcal{M}$ denote the set of maximal nilpotent orbits in $WF(\pi)$ with respect to the* closure ordering *$O \leq O'$ if $O \subset \overline{O'}$. Then $\mathcal{M}$ is the the set of maximal orbits such that*

$$\mathrm{Hom}_G(\Gamma_O, \pi^*) \neq 0,$$

and for each $O \in \mathcal{M}$, $c_O(\pi) = \dim \mathrm{Hom}_G(\Gamma_O, \pi^*)$.

In particular, the spaces $\mathrm{Hom}_G(\Gamma_O, \pi^*)$ above must be finite dimensional. In a recent paper of Gomez, Gourevitch, and Sahi the authors cleverly deform the gGGr construction with certain well-chosen "degenerate" Gelfand-Graev representations to show the following stronger result for $G = GL_n(\mathbb{F})$:

Theorem 6.3. *(Gomez, Gourevitch, and Sahi [24]) Let π be an irreducible, smooth, admissible representation of $GL_n(\mathbb{F})$, and let O be a nilpotent orbit in $\mathfrak{g}$. Then* $\mathrm{Hom}_G(\Gamma_O, \pi^*)$ *is nonzero precisely when $O \subset WF(\pi)$.*

Since the dominance order on partitions gives the same partial ordering on nilpotent orbits in $\mathfrak{gl}_n$ as the closure order, the above result can be viewed as a kind of "upper triangularization theorem" in the same vein as Andrews and Thiem: if π is an irreducible, smooth, admissible representation of $GL_n(\mathbb{F})$ such that $\mathrm{Hom}_G(\Gamma_{O_\lambda}, \pi^*) \neq 0$, then $\mathrm{Hom}_G(\Gamma_{O_\mu}, \pi^*) \neq 0$ for all $\mu \leq \lambda$, meaning representations in Theorem 6.2 control most of the relevant structure of gGGrs for $GL_n(\mathbb{F})$.

6.3 Functionals Associated to gGGrs

In Chapter 5 we discussed the left $\mathcal{H}$-module structure associated to a unique functional. Recall that the uniqueness of the functional ensured that the intertwiners $\mathcal{A}_s$ acted by the "magic factors" $b_s(\mathcal{L})$ on the image of the functional, and this in turn determined a left $\mathcal{H}$-module structure on the image of the functional $\mathcal{L}$. Here we investigate non-unique analogues of this theory. We hope to associate non-unique model spaces to R-linear maps $\mathcal{L} : \imath_B^G R \to V_R$ where V_R is a finite rank R-module, and identify when the map $\mathcal{L}$ give a map of left $\mathcal{H}$-modules.

Generalized Gelfand-Graev representations provide a useful source of examples to investigate these phenomena. We observe that if $P = LU$ is any parabolic subgroup of G and L_c is a compact subgroup of L, then $L_cU \subset G$ is exhausted by its compact open subgroups. Thus, functors of L_cU coinvariants are exact. Applying this to the case that L_c is a subgroup of the compact stabilizer of a nondegenerate character on U, we associate generalized Gelfand-Graev representations some reduced gGGrs to exact functors. By Theorem 4.10, the functor $\mathcal{J}$ of coinvariants associated to each (reduced or otherwise) p-adic gGGr gives a left $\mathcal{H}$-module map

$$\mathcal{L} : i_B^G R \to V$$

where the left $\mathcal{H}$-action on V is "the same as" the right $\mathcal{H}$-action on the gGGr Γ. We summarize our results in the following Corollary.

Corollary 6.4. *Let $\Gamma_X = \mathrm{ind}_{U_X}^G \psi$ be a gGGr, and let $\Gamma_{X,\rho}$ be a reduced gGGr for which ρ is a linear character. Then Γ_X^I and $\Gamma_{X,\rho}^I$ are finite rank as R-modules. Moreover, the functors $\mathcal{J}_{U_X,\psi}$ and $\mathcal{J}_{U_X Z_L(X),\psi\otimes\rho}$ are exact. The spaces $\mathcal{J}_{U_X,\psi}(i_B^G R)$ and $\mathcal{J}_{U_X Z_L(X),\psi\otimes\rho}(i_B^G R)$ admit left $\mathcal{H}$-actions which are "the same as" the right $\mathcal{H}$-action on Γ_X^I and $\Gamma_{X,\rho}^I$, respectively.*

In order to pave the way for a more detailed study of this situation, we briefly outline some structure theory for the $\mathcal{H}$-module structure of $\Gamma_{U_X}^I$ (or equivalently, $\mathcal{J}_{U_X,\psi}(i_B^G R)$). $\Gamma_{U_X}^I$ is spanned by functions supported on double cosets of the form $U_X uwtI$, where $u \in U/U_X$, $w \in W$, and $t \in T/T(\mathfrak{o})$. We let $f_{uwt} \in \Gamma_{U_X}^I$ denote the function supported on $U_X uwtI$ normalized so that $f(uwt) = 1$. The following Lemma establishes a direct connection to gGGrs over finite fields.

Lemma 6.5. *Let uw be a $U_{1.5}\backslash G/I$ double coset representative for which the torus component is trivial. For any function $f : G \to \mathbb{C}$ which is $G(\mathfrak{p})$-fixed, let $\overline{f}$ be the image of $f|_{G(O)}$ under the projection to $C^\infty(G(\mathbb{F}_q)) = C^\infty(G(O)/G(\mathfrak{p}))$. Let $\Gamma_X(\mathbb{F}_q)$ denote the gGGr over the finite field $\mathbb{F}_q$. We have the following:*

(i) if f_{uw} is I-fixed we may choose $u \in U'(O/\mathfrak{p})$,

(ii) the functions f_{uw} span an $\mathcal{H}_0$-module which is isomorphic to $\Gamma_X(\mathbb{F}_q)^{B(\mathbb{F}_q)}$.

Proof. First, we note that if f_{uw} is I-fixed, f_{uw} must be trivial on $G(\mathfrak{p})$. To see that we must take $u \in U(\mathfrak{o})$, we assume without loss of generality that $u = U_\beta(y)$ for some positive root β and some $y \in \mathbb{F}$. Then there exists a positive root α such that $U_\alpha \subseteq U_{1.5}$ and $U_{[\beta,\alpha]} \subseteq \operatorname{supp}\psi$ (if such an α did not exist, we could include U_β in $U_{1.5}$). If $y \notin \mathfrak{o}$,

$$U_\beta(y)U_\alpha(\mathfrak{p}) = U_{[\beta,\alpha]}(y\mathfrak{p})U_\alpha(\mathfrak{p})U_\beta(y),$$

and since $y\mathfrak{p}$ must contain the integers, f is not $U_{w^{-1}(\alpha)}(\mathfrak{p})$-fixed. Item (i) follows.

To see the second claim, let f_{uw} with $u \in G(\mathfrak{o})$. Since f is supported on $G(\mathfrak{o})$ and is trivial on $G(\mathfrak{p})$, f can be viewed as an element of $(\operatorname{ind}_{U(\mathfrak{o})G(\mathfrak{p})}^{G}(\mathfrak{o})\psi)^I$. As ψ has conductor $\mathfrak{p}$, the latter space can be identified with $\Gamma_X(\mathbb{F}_q)^{B(\mathbb{F}_q)}$. This association identifies the convolution action of $\mathbb{1}_{IwI} \in C_c^\infty(I\backslash G/I)$ with convolution by $\mathbb{1}_{B(\mathbb{F}_q)wB(\mathbb{F}_q)} \in C(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q))$. □

Recall that in the case of unique models, we are often able to identify a linear character ε of $\mathcal{H}_0$ such that the Iwahori-fixed vectors in the model space were isomorphic to $\operatorname{ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$. The next Theorem generalizes this phenomena to the generalized Gelfand-Graev models by establishing an explicit $\mathcal{H}_0$-module in Γ_X^I which contains an R-basis for Γ_X^I.

Theorem 6.6. *Let Γ_X be a generalized Gelfand-Graev representation for G, and let $\Gamma_X(\mathbb{F}_q)$ the the analogous representation for $G(\mathbb{F}_q)$. As an R-module, Γ_X^I is generated by an $\mathcal{H}_0$-module isomorphic to $\Gamma_X(\mathbb{F}_q)^{B(\mathbb{F}_q)}$.*

Proof. By Lemma 6.5, the functions f_{uw} span a copy of an $\mathcal{H}_0$-module which is isomorphic to $\Gamma_X(\mathbb{F}_q)^{B(\mathbb{F}_q)}$. By Theorem 4.7, there is an R-module isomorphism

$$\Gamma_X^I \to \mathcal{J}_U\left(\Gamma_X^I\right) = \mathcal{J}_U\left(\Gamma_X\right)^{T(\mathfrak{o})}.$$

Then the R-action on $\mathcal{J}_U(f_{uwt})$ is by translation of the index t. The result follows. □

We briefly remark that, in the case of $G = GL_n$, our study of the generalized Gelfand-Graev representations Γ_λ can proceed analogously to the finite case. In particular, if $\mathcal{H}_\mu$ denotes the parahoric Hecke algebra of G associated to partition μ, then

Γ_λ appears to have a very nice structure as an $\mathcal{H}_{\lambda^T}$-module. Early experiments suggest that it may be possible to prove an analogue of Chan and Savin's results in [12, 13] for the $\mathcal{H}_{\lambda^T}$ structure of $\Gamma_\lambda^{J_{\lambda^T}}$.

Chapter 7

Conclusion

The Iwahori-Hecke algebra illuminates significant algebraic structures associated to functionals on the unramified principal series of a reductive group over a p-adic field. We have unified the "unique model perspective" studied by Chan and Savin [12, 13] with the "unique functional perspective" of Brubaker, Bump, and Friedberg [4], and classified the type of $\mathcal{H}$-modules occurring in this theory. In the future, we would like to build a library of examples of functionals corresponding to each family in the classification results in Chapter 5. A particularly fruitful source of examples might be those arising from period integrals associated to spherical varieties, as in the work of Sakellaridis and Venkatesh [39]. Furthermore, it would be interesting to investigate which $\mathcal{H}$-stable ideals arise from "naturally occurring" unique models, and what this says about the types of unramified principal series representations that are captured by these models.

There is also significant work to be done in expanding this theory to finite multiplicity (i.e. larger than one) models, such as the generalized Gelfand-Graev representations of Kawanaka [28, 29]. Possible research questions are as follows. Is there a more precise description of the right $\mathcal{H}$-modules Γ_X^I from chapter 6 than provided in Theorem 6.6? Are there are restrictions on the structure constants associated to the left $\mathcal{H}$-modules $\mathcal{J}_{U_X,\psi}(i_B^G R)$ as there were in the finite case? What can be said about the reduced generalized Gelfand-Graev representations? Is there an analogue to the rigidity condition from Chapter 4, which gave our classification results in the finite case? These investigations have potential to greatly expand the theory of p-adic functionals and eventually inform the global theory of automorphic forms.

Appendix A

Deligne-Lusztig Classification of Irreducible Representations over Finite Fields

This chapter briefly outlines the Deligne-Lusztig classification of irreducible characters for reductive groups over finite fields alluded to in Chapters 2 and 3. This contextualizes and motivates our study of representations with B-fixed vectors in this setting. Over finite fields, this class of representations are exactly the *unipotent representations in the principal series*. Such representations originally arose in Deligne and Lusztig's classification of irreducible representations for reductive groups over finite fields. In this appendix, we review the Deligne-Lusztig classification and discuss Kawanaka's alternate parametrization of these representations using gGGrs.

Deligne-Lusztig Theory

In this section we follow Carter's exposition in [8] and [9] of the 1976 work of

Deligne and Lusztig in [18] and the 1984 book of Lusztig [32], emphasizing results over technique. For this section, let G denote a connected, reductive group *over an algebraically closed field of characteristic* p, and let σ denote the corresponding Frobenius map, so that G^σ is a reductive group over a finite field. First, we define the Deligne-Lusztig virtual characters. Let T be a maximal (not necessarily split) σ-stable torus in G, and let $\theta : T \to \mathbb{C}$ be a linear character of T. Let B be a Borel subgroup of G containing B, and let U be the unipotent radical of B. Let $L : G \to G$ denote the *Lang map*

$$L(g) := g^{-1}\sigma(g).$$

It is noteworthy that $L^{-1}(1) = G^\sigma$. Finally, let X denote the *Deligne-Lusztig variety* $L^{-1}(U)$. If $\ell \neq p$ is prime, let $\mathbb{Q}_\ell$ denote the ℓ-adic rationals. Then the left action of G^σ on X lifts to an action on the ℓ-adic cohomology groups with compact support $H^i_c(X, \overline{\mathbb{Q}}_\ell)$. The *Deligne-Lusztig virtual character* $R_{T,\theta} : G^\sigma \to \mathbb{C}$ is defined as

$$R_{T,\theta}(g) := \sum_{i\geq 0}(-1)^i \operatorname{Tr}(g, H^i_c(X, \overline{\mathbb{Q}}_\ell)_\theta),$$

where T^σ acts on $H^i_c(X, \overline{\mathbb{Q}}_\ell)_\theta$ by θ. The values $R_{T,\theta}(g)$ are algebraic integers over $\mathbb{Q}$, so this map can be understood as complex-valued. As the notation suggests, $R_{T,\theta}$ is independent of the choice of Borel subgroup B, although this is not trivial. $R_{T,\theta}$ is a *virtual character* since it is a $\mathbb{Z}$-linear combination of characters on G^σ, and so might not be the trace of any representation of G^σ. One can view the virtual characters $R_{T,\theta}$ as generalizations of the principal series, as illustrated in the following theorem:

Theorem A.1. *Let the notation be as above. Then*

(a) If T is a maximally split, $R_{T,\theta}$ agrees with the corresponding principal series:

$$R_{T,\theta} = \operatorname{ind}_{B^\sigma}^{G^\sigma} \theta_B,$$

where θ_B is θ viewed as a character of B.

(b) If θ is in general position (i.e. $\theta \neq \theta^w$ for any Weyl group element w), then $\pm R_{T,\theta}$

is irreducible.

(c) *If T' is another σ-stable maximal torus which is not G^σ-conjugate to T, then the inner product of characters $\langle R_{T,\theta}, R_{T',\theta}\rangle = 0$.*

(d) *Each irreducible character of G^σ occurs as a summand in some virtual character $R_{T,\theta}$.*

Since the class functions $R_{T,\theta}$ are only virtual characters, $\langle R_{T,\theta}, R_{T',\theta}\rangle = 0$ does not immediately imply that $R_{T,\theta} = R_{T',\theta}$. To better understand when $R_{T,\theta}$ and $R_{T',\theta'}$ share summands, we need to study more subtle properties of the pairs (T, θ). We say two pairs (T, θ) and (T', θ') are *geometrically conjugate* if exists some $n \in \mathbb{N}$ such that, for some $g \in G^{\sigma^n}$, $T^g = T'$ and $(\theta \circ N)^g = (\theta' \circ N)$, where $N : \mathbb{F}_{q^n} \to \mathbb{F}_q$ is the norm map.

At this point, we must add an additional assumption that the center of G is connected. This is necessary to streamline the following discussion of the Jordan decomposition of characters, although many of the classification results can be stated without this assumption.

Theorem/Definition A.2. *Let the notation be as above.*

(a) *There is a unique geometric conjugacy class containing pairs of the form $(T, 1)$. An irreducible character χ of G^σ which occurs in some $R_{T,1}$ is called* unipotent*. The collection of unipotent characters of G^σ is denoted $\hat{G}^\sigma_u$.*

(b) *For each geometric conjugacy class κ of pairs (T, θ), the class function*

$$\sum_{\substack{(T,\theta)\in\kappa \\ \text{mod } G^\sigma}} \frac{R_{T,\theta}}{\langle R_{T,\theta}, R_{T,\theta}\rangle}$$

is an irreducible character of G^σ up to sign. The irreducible characters obtained this way are called semisimple*, and they are distinct for distinct geometric orbits. Assuming p is a good prime, these are exactly the irreducible characters of G with order prime to p. The collection of semisimple characters of G^σ is denoted $\hat{G}^\sigma_s$,* *and if χ is any irreducible character of G^σ, $\hat{G}^\sigma_s(\chi)$ denotes the unique semisimple character of G^σ sharing a geometric conjugacy class with χ.*

By Theorem A.1, if T is maximally split, $R_{T,1} = \mathrm{ind}_{B^\sigma}^{G^\sigma} 1$ for some Borel subgroup B^σ of G^σ containing T. Independently, we know $(\mathrm{ind}_{B^\sigma}^{G^\sigma} 1)^{B^\sigma}$ is the regular representation of the Hecke algebra $\mathcal{H}_0$ of G^σ. The following result summarizes the connection between irreducible $\mathcal{H}_0$-modules and unipotent representations.

Corollary A.3. *There is a bijection*

$$\left\{\begin{array}{c}\textit{unipotent representations}\\ \textit{in the principal series of } G^\sigma\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\textit{irreducible modules}\\ \textit{of } \mathcal{H}_0\end{array}\right\}.$$

In type A, each unipotent representation occurs in the principal series. Thus, the above is a bijection between irreducible $\mathcal{H}_0$-modules and all unipotent representations in this case. Unipotent representations which do not occur in the principal series are called *cuspidal unipotent representations*.

The terminology 'unipotent' and 'semisimple' can be viewed in analogy to the usual notions of unipotent and semisimple elements. In G^σ, the unipotent elements are those with order some power of p, and the semisimple elements are those with order prime to p. These concepts lead to the following:

Theorem A.4. *(The Jordan Decomposition of Characters) Let the notation be as above, and let G^* be the dual group to G with Frobenius map σ^*. There is a bijection between semisimple characters of G^σ and semisimple conjugacy classes of $G^{*\sigma^*}$. Letting $\phi(\chi_s)$ denote some chosen element in the semisimple conjugacy class of $G^{*\sigma^*}$ associated to a semisimple character χ_s, we have a bijection*

$$\{\textit{irreducible characters } \chi \textit{ of } G^\sigma\} \leftrightarrow \{\textit{pairs } (\chi_s, \chi_u) \textit{ where } \chi_s \in \hat{G}^\sigma_s \textit{ and } \chi_u \in \hat{C}_{G^{*\sigma^*}}(\phi(\chi_s))_u\}.$$

Moreover, if $\chi \mapsto (\chi_s, \chi_u)$ under this pairing, then $\chi_s = \hat{G}^\sigma_s(\chi)$ and $\chi(1) = \chi_s(1)\chi_u(1)$.

The groups $\hat{C}_{G^{*\sigma^*}}(\phi(\chi_s))$ will always be reductive, and so the above theorem indicates that it is sufficient to separately study semisimple and unipotent characters of finite reductive groups. As the semisimple characters are given by an explicit formula from part (b) of Theorem A.2, it is sufficient to understand the unipotent characters. For a character ϕ of W, define a character R_ϕ of G^σ by

$$R_\phi := \frac{1}{|W|}\sum_{w\in W}\phi(w)R_{T_w,1},$$

where T_w is a torus corresponding to the conjugacy class of w.[1] One may define an equivalence relation on the unipotent characters of G^σ by relating two characters who appear in the same sum R_ϕ. These equivalence classes are called *families of unipotent characters* of G^σ. By associating a *special unipotent* element of G^σ to each such family, one can define a group $\overline{A}(u)$ which is a quotient of the component group of the centralizer of u. The unipotent representations in the family associated to u are parametrized by pairs (x,ϕ), where $x\in\overline{A}(u)$ and $\phi\in C_{\overline{A}}(u)^\wedge$ up to the action of $\overline{A}(u)$.

Theorem A.5. *(The Deligne-Lusztig Parametrization) The irreducible characters of G^σ are parametrized by pairs (χ_s,χ_u), where χ_s is a semisimple character of G^σ and χ_u is a unipotent character associated of a subgroup of $G^{*\sigma^*}$ determined by χ_s as in the Jordan decomposition of characters. Since each such subgroup is reductive, its unipotent characters are determined by triples (u,x,ϕ), where u is special unipotent, $x\in\overline{A}(u)$, and $\phi\in C_{\overline{A}(u)}(x)^\wedge$.*

One can of course make this more precise, and Lusztig adapted this theory to work in the case where G does not have a connected center. That said, the above theorem serves our purposes by highlighting the similarities between Kawanaka's conjecture and the work of Deligne and Lusztig.

[1] If T_0 is a maximally split torus, all maximal tori are some G-conjugate of T_0. In particular, all σ-stable maximal tori of G are of the form T_0^g for some g such that $L(g)\in N_G(T_0)$. Projecting onto the Weyl group yields a bijection between G^σ conjugacy classes of maximal tori and conjugacy classes of W (see section 3.3 of [8]).

One problem with Lusztig's parametrization is that all of the "moving parts" - the virtual characters, the unipotent representations, etc. - are difficult to define, making them tough to work with. In his paper [29], Kawanaka was able to use his (non-refined) generalized Gelfand-Graev representations to give models for both the Deligne-Lusztig virtual characters and the unipotent representations appearing in Lusztig's classification. In particular, Kawanaka showed character formulas for gGGrs in terms of the virtual characters $R_{T,\theta}$. Furthermore, Kawanaka showed the following:

Theorem A.6. *(Theorem 2.4.1 of [29]) Let χ be an irreducible character of G, and let $\mathcal{O}_\chi$ be the associated nilpotent orbit under the above map. Then*

(a) there exists some $u \in \mathcal{O}_\chi$ such that $\langle \Gamma_u, \chi \rangle \neq 0$, and

(b) for all nilpotent orbits $\overline{\mathcal{O}} \not\subset \overline{\mathcal{O}}_\chi$ and all $v \in \mathcal{O}$, $\langle \Gamma_v, \chi \rangle = 0$.

In particular, for $G = GL(n)$, the closure order on nilpotent orbits agrees with the dominance order on partitions, so the above can be viewed as a natural precursor to Theorem 3.3. We used the following conjecture of Kawanaka as a guiding principle for much of this paper:

Conjecture A.7. *(Conjecture 2.4.5 of [29]) Let $\mathcal{F}_\mathcal{O}$ be a family of unipotent representations as above. Then there is a unique labeling $\rho_{(x,\phi)}$ of elements of $\mathcal{F}_\mathcal{O}$ such that*

$$\left\langle \Gamma_{(x,\phi)}, \rho_{(x',\phi')} \right\rangle = \delta_{(x,\phi),(x',\phi')}.$$

In the above, $\Gamma_{(x,\phi)} := \Gamma_{u_x,\phi}$, where u_x is a unipotent element associated to x and ϕ is inflated to a representation of $Z_L(X)$.

As mentioned above, for $GL(n)$ the group $A(u)$ will always be trivial, so the above reduces to the statement that $\left\langle \Gamma_{\lambda,\rho_{triv}}, \pi_{\lambda^T} \right\rangle = 1$ which was shown in Corollary 3.9.

Index of Notation

Φ^- - negative roots, 41

χ_u - universal character $\mathbb{C}[X^*] \to \mathbb{C}[T/T(\mathfrak{o})]$, 4

δ_H - Haar measure on a group H, 44

ind - smooth and compactly supported induction, 3

$\tilde{H}_\lambda$ - dominant semisimple element of $\mathfrak{gl}_n$ associated to partition λ, 12

$\mathbb{F}$ - finite or p-adic field, 1, 41

$\mathbb{1}_X$ - characteristic function of X, 54

$\mathcal{A}_w$ - normalized intertwiner on $i_B^G R$ associated to $w \in W$, 45

$\mathcal{H}$ - Iwahori-Hecke algebra, 42

$\mathcal{H}_0$ - finite Hecke algebra, 26

$\mathcal{H}_K$ - Hecke algebra of K-fixed vectors, 43

$\mathcal{J}_{H,\psi}$ - functional of H, ψ coinvariants, 48

$\mathcal{O}$ - nilpotent orbit in $\mathfrak{u}$ or unipotent orbit in U, 16

$\mathfrak{b}$ - Lie algebra of B, 8

$\mathfrak{g}$ - Lie algebra of G, 8

$\mathfrak{o}$ - ring of integers of a non-Archimedean local field, 4, 41

$\mathfrak{p}$ - maximal ideal in $\mathfrak{o}$, 4, 41

$\mathfrak{u}$ - Lie algebra of u, 8

$\mathfrak{u}_k$ - sum of $i \geq k$-eigenspaces of a semisimple element $H \in \mathfrak{g}$, 16

$\mathfrak{u}_{1.5}$ - maximal subordinate subalgebra of $\mathfrak{u}$ with respect to $X^\vee \in \mathfrak{u}^*$, 17

ϖ - uniformizer in $\mathfrak{o}$, 4

$b_s(\mathcal{L})$ - "magic factor" from functional equation associated to $\mathcal{A}_s$, 68

e_α - α root subspace of $\mathfrak{u}$, 8

i - smooth and compactly supported parabolic induction, 4, 41

q - residue characteristic of $\mathfrak{o}/\mathfrak{p}$, 4, 41

gGGr - generalized Gelfand-Graev representation, 3

rgGGr - reduced generalized Gelfand-Graev representation, 3

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